Marc Yor
Editor

Aspects of Mathematical Finance

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Contents

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Introduction: Some Aspects of Financial Mathematics

M. Yor

The articles in this volume summarize the lectures given by the authors on February 1, 2005, at an open conference held at the Académie des Sciences, quai Conti, on Financial Mathematics, a field in constant development since the end of the 1970s.

The aim is to make these articles accessible to a wide audience belonging to scientific horizons; indeed, this half-day conference was attended by many young postgraduates in disciplines such as economics, management, and applied mathematics, along with many members of the different sections of the Académie.

Among the reasons to get interested in financial mathematics, the following is one: who has never wondered, looking at the financial pages of a newspaper, displaying the erratic evolutions of quotations on the Stock Exchange, if these were not "governed" by some models, likely to be probabilistic?

This question was at the heart of the studies conducted by Louis Bachelier, particularly in his famous thesis (1900), and he answered the above question in terms of Brownian Motion. His remarkable results, however, remained in a kind of scientific *limbo* for almost 75 years, until Samuelson "corrected" Bachelier (in 1965) by replacing the Brownian Motion by its exponential, and the famous Black-Scholes formula began (in 1973) to play an essential role in the computation of option prices.

Since the 1980s, we have witnessed the explosion of probabilistic models, along with financial products, each in turn becoming more and more complex.

All this technology, which now forms an important part of financial engineering, exists only because some mathematical concepts both simple and universal allow building a "theory of the laws of markets," based on principles such as *the prices across time of an uncertain asset having the probabilistic structure of a fair game, that is to say of a martingale*. From this concept, little by little was built the entire theory of stochastic processes, which is a pillar on which the mathematical theory of "arbitrage" was developed by Delbaen and Schachermayer.

The six articles in this volume present the following main topics.

H. Föllmer introduces the main probabilistic notions: martingales, martingale measures, risk measures, etc., which have been used and developed during the last 30 years to describe and explain the behavior of financial markets.

W. Schachermayer gives an introduction to the theory of “arbitrage,” proceeding progressively to a mathematical sophistication of his arguments.

P. Barrieu and N. El Karoui present a global overview of the existing interactions between mathematics and finance by emphasizing the different aspects of research in financial risk management.

H. Geman shows that even if the Brownian model (as used by Bachelier (1900) or Black–Scholes (1973), or ever since) is the object of many critical statements, it is nevertheless present in several models, up to a time change, which may even present time jumps. These time changes can be related to different changing speeds between certain financial markets.

D. Lamberton shows how partial differential equations intervene in the study of financial models and presents the numerical methods used to compute the pricing and hedging of options.

E. Gobet, G. Pagès, and M. Yor present a historical evaluation, corresponding to the last 30 years of the evolution of financial markets. Then they explain how applied mathematics, and probability theory in particular, come into play in this evolution. They finally describe in detail the existing qualifying curricula available in French universities for future “quants” in the banking and insurance industries.

These six contributions can be read independently, although they deal with the same subject. As a whole, they present a fairly complete overview of the existing interactions between mathematics and finance during the last 30 years.

This booklet, in particular the English version, benefited from the help of Monique Jeanblanc and Dilip Madan, to whom I am very grateful. Special thanks to Kathleen Qechar for her translation of some of the articles from French to English.

I personally, and also together with the other authors, thank Mr Jean Dercourt, Secrétaire perpétuel de l’Académie des Sciences, as well as Mrs. Bonfils, Martin, and Morand for their warm support and all the efforts they made in order that this Conference day of February 1, 2005 be a success.

I also salute the memory of Mr. Hubert Curien, who so kindly agreed to take the chair of this session, and who passed away a few days later. This book is gratefully dedicated to him.

Financial Uncertainty, Risk Measures and Robust Preferences

H. Föllmer

Abstract The author introduces the main probabilistic concepts: martingales, martingale measures, risk measures ... which have been used and developed during the last thirty years to describe and explain the behavior of financial markets.

1 The Role of Mathematics in Finance

When one takes a look at the European financial markets as described by the fluctuation of the EuroStoxx index during the last twelve months, one sees a very irregular trajectory. When one observes it on the scale of a single day, this irregular and unpredictable character of the path does not change. Even in the very short term, there is therefore no way of making a safe extrapolation.

Given this uncertainty, the market offers many possibilities of placing a financial bet on the future behaviour of the index. For instance, one can buy or sell a contract which gives the right to the value of the index six months later. In fact there is an increasing number of derivative products such as options and certificates which allow one to configure arbitrarily complex financial bets.

What is the role of mathematics, and in particular of probability theory, in this financial context? As with games of chance, mathematics cannot help to win a financial bet. In particular, it does not offer any means to compute in advance the outcome. On the other hand, mathematics may help to understand the nature of a given bet by providing methods to decide whether the bet is advantageous, fair, or acceptable, and to quantify its risk. Moreover, nowadays mathematics intervenes more and more in the construction of such bets, that is, in the design of new financial products.

A choice among the great variety of financial instruments depends upon the investor's preferences, and in particular on his or her tolerance towards the downside risk. We shall see how mathematics may help to specify preferences in a quantitative manner and, then, to determine the optimal structure of the bet according to these preferences.

2 Probabilistic Models

The mathematical analysis of risks and preferences is carried out in the context of a probabilistic model. Such a model specifies first the set Ω of all scenarios which we are ready to consider. Here we focus on the behaviour of a unique uncertain asset such as the EuroStoxx index between today and some time T in the future, and we exclude the occurrence of sudden jumps. Thus a scenario can be described by a continuous function ω on the interval $[0,T]$, and we shall denote by $S_t(\omega) = \omega(t)$ the value of the asset at time t in this scenario.

In order to arrive at a probabilistic model, one should then choose a probability measure P on the set Ω of all scenarios. Once this choice has been made, one can now make predictions, not in the sense of guessing the future scenario, but by assigning probabilities to sets of scenarios and expectations to monetary quantities whose value may vary from scenario to scenario. If the value of a financial position at some future time t is described by a function $V_t(\omega)$ depending on the scenario up to time t, one can thus consider V_t as a random variable on the probability space (Ω,P) and compute its expectation $E[V_t]$. More generally, we denote by $E_s[V_t]$ the conditional expectation of the future value V_t, given that we have observed the scenario up to some time s prior to time t.

Which probability measure should we choose? In 1900, Louis Bachelier introduced Brownian motion as a model of price fluctuations on the Paris stockmarket. A rigorous construction of the corresponding measure P on the set Ω of all continuous functions on the interval $[0,T]$ was given by Norbert Wiener in 1923. It is characterised by the property that the increments $S_t - S_s$, considered as random variables on the probability space (Ω,P), are Gaussian with means and variances proportional to the length $t-s$ of the time interval, and that they are independent for disjoint intervals. If this construction is carried out on a logarithmic scale, one obtains *geometric Brownian motion*, by now a standard model for the price fluctuation of a liquid financial asset, which was proposed by P.A. Samuelson in the 1960s. Moreover, by allowing a change of clock which may depend on the scenario, one opens the door to a large variety of probabilistic models, including the solutions of stochastic differential equations. Here we are restricting the discussion to a single index, but of course we could also model the simultaneous fluctuation of several assets. Furthermore, one may want to analyse the microstructure of the market by taking into account the interactive behaviour of heterogeneous agents, and then models become extremely complex.

Thus there is a vast number of modelling choices, and an entire industry is aiming to calibrate certain model classes with the help of econometric and statistical methods. In this exposition, however, we do not discuss such modelling issues in detail. We shall rather focus on some theoretical principles related to the idea of *market efficiency* and to the notion of *financial risk*.

3 Market Efficiency and Martingale Measures

In its strongest version, the hypothesis of market efficiency demands that the measure P be such that the successive prices S_t of the uncertain asset have the probabilistic structure of a fair game. In mathematical terms, this means that the predictions of future prices coincide with the present prices, i.e.

$$E_s[S_t] = S_s$$

for every time t and every time s prior to time t. Equivalently, one may write

$$E_s[S_t - S_s] = 0,$$

which means that the conditional expectation of the net gain $S_t - S_s$ knowing the scenario up to time s is always zero. In this case, the stochastic process $(S_t)_{0\le t\le T}$ is said to be a *martingale* with respect to the measure P, and P is called a *martingale measure*.

For now, let us suppose that P is indeed a martingale measure. In this case, a fundamental theorem in martingale theory due to J.L. Doob implies that there are no advantageous investment strategies. More precisely, let us consider a strategy which starts with an initial capital V_0 and divides at each time the available capital between the uncertain asset and a safe asset with interest rate r. To simplify we suppose that r is equal to 0 (in fact we have already implicitly done so in our definition of the martingale measure). Let us denote by $V_T(\omega)$ the value generated by the strategy up to time T as a function of the scenario ω. Allowing for some realistic restrictions on the strategy, one obtains the equation

$$E[V_T - V_0] = 0.$$

for the random variable V_T. Thus, there is no way to design a strategy such that the expectation of the net benefit be positive. Clearly, this conclusion is not shared by the great majority of financial analysts and consultants. In fact there are good reasons to believe that the strong version of the hypothesis of market efficiency is too rigid.

Let us therefore turn to a much more flexible way of formulating the efficiency hypothesis for financial markets. Here we admit the existence of advantageous strategies, but we exclude the possibility of a positive expected gain without any downside risk. More precisely, one requires that the probability measure P does not admit any strategy which on the one hand is advantageous in the sense that

$$E[V_T] > V_0,$$

and on the other hand is safe in the sense that the probability of a loss is zero, i.e.

$$P[V_T < V_0] = 0.$$

Under this relaxed version of the efficiency hypothesis, the measure P is not necessarily a martingale measure. But the mathematical analysis shows that there must exist a martingale measure P^* which is equivalent to P in the sense that these two measures give positive weight to the same sets of scenarios; see Schachermayer [17] or Delbaen and Schachermayer [4].

From now on, we shall therefore assume that there exists a martingale measure P^* which is equivalent to P, and we shall denote by $\mathcal{P}^*$ the class of all these measures.

4 Hedging Strategies and Preferences in the Face of Financial Risk

In the case of simple diffusion models such as geometric Brownian motion, the equivalent martingale measure P^* is in fact unique. In this case there exists a canonical way of computing the prices of all the derivative products of the underlying asset. Let us denote by $H(\omega)$ the value of such a product at time T as a function of the scenario ω. The uniqueness of the martingale measure P^* implies that there exists a strategy with initial capital

$$V_0 = E^*[H]$$

such that the value $V_T(\omega)$ generated up to time T coincides with $H(\omega)$ for every scenario ω outside of some set of probability zero. Thus, this *hedging strategy* allows a perfect replication of the financial product described by the random variable H. The initial capital it needs is given by the expectation $E^*[H]$ of the random variable H with respect to the martingale measure P^*. This expectation is also the canonical price of the product. Indeed, every other price would offer the possibility of a gain without risk. If, for instance, the price were higher than $E^*[H]$ then one could sell the product at that price and use the sum $E^*[H]$ to implement the hedging strategy which allows to pay what is needed at the final time T. The difference between the price and the initial cost $E^*[H]$ would thus be a gain without risk, and such a *free lunch* is excluded by our assumption of market efficiency.

The situation becomes much more complicated if the equivalent martingale measure is no longer unique. One can still construct a hedging strategy which makes sure that the final value $V_T(\omega)$ is at least equal to the value $H(\omega)$ for every scenario outside of a set with zero probability. But the initial capital which would be needed is now given by

$$V_0 = \sup E^*[H]$$

where the supremum is taken over all measures P^* in the class $\mathcal{P}^*$ of equivalent martingale measures. From a practical point of view, this sum is typically too high.

A more pragmatic approach consists in abandoning the aim of a perfect hedge and in accepting a certain *shortfall risk*. It is at this point that the investor's preferences come into play. In order to quantify the shortfall risk of a strategy with final outcome V_T, the investor may for instance take the probability

$$P[V_T < H]$$

of generating a value smaller than the amount H which will be needed in the end. Or one could define the shortfall risk as an expectation of the form $E_P[l(H - V_T)]$ where l is an increasing and convex loss function. Once this choice is made, one can then determine the strategy which minimises the shortfall risk under the constraint of a given initial capital V_0; see Föllmer and Leukert [6].

In specifying preferences related to risk, we have just started to use explicitly the probability measure P. Let us point out that this was not the case in the preceding discussion which only made use of the class $\mathcal{P}^*$ of equivalent martingale measures. In practice the choice of a probability measure, and therefore of a systematic procedure of making probabilistic predictions, is usually not obvious. As Alan Greenspan said some time ago:

Our ability to predict is limited. We need some humility.

How can we rephrase this maxim of humility in mathematical terms? We shall describe a way of doing just that by explaining a recent approach to the quantification of financial risk which takes model uncertainty explicitly into account, and which does not rely on a specific probabilistic model to be fixed beforehand.

5 Risk Measures and Model Uncertainty

Let us consider a financial position, and denote by $X(\omega)$ the net result at the end of the period $[0,T]$ for the scenario ω. The position is therefore described by a function X on the set Ω of all scenarios. We shall now describe a way of quantifying the risk $\rho(X)$ of such a position in monetary terms.

Suppose that in the space $\mathcal{X}$ of all financial positions we have singled out a subset $\mathcal{A}$ of positions which are judged to be 'acceptable', for example from the point of view of a supervising agency. Let us write $X \geq Y$ if $X(\omega) \geq Y(\omega)$ for every scenario ω. It is then natural to assume that X is acceptable as soon as there exists an acceptable position Y such that $X \geq Y$. Let us define the risk $\rho(X)$ as the minimal amount of capital such that the position X becomes acceptable when this amount is added and invested in a risk free manner. In mathematical terms,

$$\rho(X) = \inf\{m | X + m \in \mathcal{A}\},$$

where the infimum is taken over the class of all constants m such that the new position $X + m$ is acceptable. It follows that ρ is a functional defined on the class $\mathcal{X}$ of all positions such that $\rho(X + m) = \rho(X) - m$ for every constant m. Moreover, the risk decreases when the outcome increases, i.e. $\rho(X) \leq \rho(Y)$ if $X \geq Y$. Let us call such a functional ρ on $\mathcal{X}$ a *monetary risk measure*.

Let us suppose, for now, that a probability measure P is given on the set Ω of all scenarios. A classical way of quantifying the risk of a position X, now considered as a random variable on the probability space (Ω, P), consists in computing its variance. But the definition of the variance, which is natural as a measure of measurement risk in the theory of errors, does not provide a monetary risk measure. In particular the symmetry of the variance does not capture the asymmetric perception of financial risk: what is important in the financial context is the *downside risk*. One might therefore decide to find a position X acceptable if the probability of a loss is below a given threshold, i.e. $P[X<0] \leq \beta$ for a fixed constant β. The resulting risk measure, called *Value at Risk* at level β, takes the form

$$\mathrm{VaR}(X) = \inf\{m | P[X+m<0] \leq \beta\} = -\sup\{x | P[X<x] \leq \beta\}.$$

Thus, it is given by a quantile of the distribution of X under the probability measure P. Value at Risk is a monetary risk measure, and it is very popular in the world of Finance. It has, however, a number of shortcomings. In particular the class $\mathcal{A}$ of acceptable positions is not convex. In financial terms this means that there are situations where *Value at Risk* may penalise the diversification of a position, even though such a diversification may be desirable.

If one does not want to discourage diversification then one should insist on the convexity of the set $\mathcal{A}$. In this case, the functional ρ is a *convex risk measure* in the sense of Frittelli and Gianin Rosazza [9] and Föllmer and Schied [7, 8]. If the class $\mathcal{A}$ is also a cone then we obtain a *coherent risk measure* as defined by Artzner, Delbaen, Eber and Heath [1] in a seminal paper which marks the beginning of the theory of risk measures. Note, however, that the cone property of the class $\mathcal{A}$ implies that a position X remains acceptable if one multiplies it by a positive factor, even if this factor becomes very large. Various problems arising in practice, such as a lack of liquidity as one tries to unwind a large position, suggest a more cautious approach. This is why we only insist on convexity.

Here is a first example. Suppose that we have chosen a probabilistic model of the financial market. Let us say that a position X is acceptable if one can find a trading strategy with initial capital 0 and final value V_T such that the value of the combined position $X+V_T$ is at least equal to 0 for all scenarios outside of a set with probability 0. If we impose no constraints on the trading strategies then the resulting risk measure will be coherent, and it takes the form

$$\rho(X) = \sup E^*[-X]$$

where the supremum is taken over the class $\mathcal{P}^*$ of all equivalent martingale measures. Convex restrictions for the trading strategies will lead to convex rather than coherent risk measures. This is another motivation to pass from coherent to convex risk measures.

Let us consider a second example. Suppose that the investor's preferences are represented by a utility functional U on the space $\mathcal{X}$ in the sense that a position X is preferred to a position Y if and only if $U(X) > U(Y)$. Then the investor may find a position X acceptable as soon as the value $U(X)$ does not fall below a given level. If the functional U takes the classical form of an *expected utility*

$$U(X) = E_P[u(X)],$$

defined with the help of a probability measure P and some concave and increasing utility function u, then one obtains a convex risk measure. For example, the utility function $u(x) = 1 - \exp(-\gamma x)$ induces, up to an additive constant, the *entropic* risk measure

$$\rho(X) = \frac{1}{\gamma} \log E_P[\exp(-\gamma X)] = \sup(E_Q[-X] - \frac{1}{\gamma} H(Q|P)).$$

Here, the supremum is taken over the class of all probability measures Q on the set Ω of all scenarios, and $H(Q|P)$ denotes the relative entropy of Q with respect to P, defined by $H(Q|P) = E_Q[\log \varphi]$ if Q admits the density φ with respect to P and by $H(Q|P) = +\infty$ otherwise.

In both examples, the risk measure is of the form

$$\rho(X) = \sup(E_Q[-X] - \alpha(Q)) \tag{1}$$

where the supremum is taken over all probability measures Q on the set Ω of all scenarios, and where α is a penalisation function which may take the value $+\infty$. In the first example one has $\alpha(Q) = 0$ if Q is an equivalent martingale measure and $\alpha(Q) = +\infty$ otherwise. In the second example, $\alpha(Q)$ is equal to the relative entropy $H(Q|P)$ divided by the parameter γ.

In fact, the representation (1) gives the general form of a *convex risk measure*, granting some mild continuity conditions. This follows by applying a basic duality theorem for the Legendre–Fenchel transformation to the convex functional ρ; see, for example, Delbaen [3], Frittelli and Gianin Rosazza (2002) or Föllmer and Schied [8]. In order to compute the value $\rho(X)$ from the representation (1), it obviously suffices to consider the class $\mathcal{Q}$ of all probability measures Q such that $\alpha(Q)$ is finite. In the coherent case, the penalisation function only takes the two values 0 and $+\infty$. Thus, the representation (1) reduces to the form

$$\rho(X) = \sup E_Q[-X] \tag{2}$$

where the supremum is taken over the class $\mathcal{Q}$, and this is the general form of a *coherent* risk measure.

Thus, the computation of the risk of a position X reduces to the following procedure. Instead of fixing one single probabilistic model, one admits an entire class $\mathcal{Q}$ of probability measures on the set of all scenarios. For every measure Q in this class, one computes the expectation $E_Q[-X]$ of the loss $-X$. But one does not deal with these measures in an equal manner: some may be taken more seriously than others, and this distinction is quantified by the subtraction of the value $\alpha(Q)$. Once this subtraction is made, one then considers the least favourable case amongst all models in the class $\mathcal{Q}$, and this defines the monetary risk $\rho(X)$ of the position X.

6 Analogies with the Microeconomic Theory of Preferences

In the financial context, the preferences of an investor are described by a partial order on the space $\mathcal{X}$ of all financial positions. Under some mild conditions, such a partial order admits a numerical representation in terms of some utility functional U on $\mathcal{X}$. This means that the financial position X is preferred to the position Y if and only if $U(X) > U(Y)$. In the classical paradigm of *expected utility*, there is one single probability measure P on the set of all scenarios which is given a priori, and the functional U is of the form

$$U(X) = E_P[u(X)] \tag{3}$$

with some concave and increasing utility function u, as we have seen it already in the second example above. In this classical setting, the value $U(X)$ depends only on the distribution μ of the position X, considered as a random variable on the probability space (Ω, P), since it can be expressed as the integral

$$U(X) = \int u d\mu \tag{4}$$

of the utility function u with respect to μ. One may thus consider preferences as a partial order on the class of probability distributions on the real line. J. von Neumann and O. Morgenstern have characterised, via some axioms which formalise a rather strict notion of *rationality*, those partial orders on the class of distributions which admit a representation of the form (4) in terms of an implicit utility function u.

More generally, and without fixing a priori a probability measure P on the set of all scenarios, L. Savage, R. Aumann (Nobel prize 2005) and others have specified axioms for a partial order on the class $\mathcal{X}$, which allow us to reconstruct from the given preferences an implicit probability measure P and an implicit utility function u which yield a numerical representation of the form (3).

It is well known, however, that the paradigm of expected utility is not compatible with a number of empirical observations of people's behaviour in situations involving uncertainty. In order to take into account such findings, and in particular certain 'paradoxes' due to D. Ellsberg, M. Allais (Nobel prize 1988) and D. Kahneman (Nobel prize 2002), Gilboa and Schmeidler [10] have proposed a more robust notion of rationality. In their axiomatic setting, the utility functional takes the form

$$U(X) = \inf E_Q[u(X)] \tag{5}$$

where the infimum is taken over a whole class $\mathcal{Q}$ of probability measures Q on the set of scenarios. Such *robust preferences* are not determined by a unique probabilistic model. Instead, they involve a whole class of probabilistic model, and thus they take explicitly into account the model uncertainty which is typical in real world situations. Up to a change of sign, there is an obvious analogy between coherent risk measures and robust preferences characterised by the representation (5). The theory of convex risk measures suggests to go one step further and to consider a modification of the Gilboa–Schmeidler axioms which leads to a functional of the form

$$U(X) = \inf(E_Q[u(X)] + \alpha(Q)) \tag{6}$$

where α is a penalisation function, in analogy with the representation (1) of a convex risk measure. In fact, the preferences on $\mathcal{X}$ which do admit a numerical representation of the form (6) have recently been characterised by Maccheroni, Marinacci and Rustichini [15].

7 Optimisation Problems and Robust Projections

In view of the large variety of financial bets which are available due to the increasing number of derivative products, it is not easy to make a choice. Let us now consider the mathematical problem of computing an optimal financial position X, given an initial capital V_0 as well as the investor's preferences.

Assuming that preferences are represented via a functional U on the space $\mathcal{X}$ of financial positions, the problem consists in maximising the value $U(X)$ under the condition that the position X can be financed by the available capital V_0. This means that one can find a trading strategy with initial value V_0 such that the resulting final value V_T is at least equal to X. It can be shown that this feasibility condition amounts to the constraint

$$\sup E^*[X] \le V_0$$

where the supremum is taken over the class $\mathcal{P}^*$ of the equivalent martingale measures. We shall now assume that U is a functional with robust utility of the form (5), defined by a class $\mathcal{Q}$ of probability measures which are equivalent to P. Thus, our optimisation problem involves the two classes of measures $\mathcal{P}^*$ and $\mathcal{Q}$.

Let us first consider the case where each one of these two classes only contains one single element. In particular, the preferences are of the form (3) for one single measure Q, and there is a unique equivalent martingale measure P^*. In this classical case, the solution to the optimisation problem is given by

$$X = (u')^{-1}\big(\lambda\varphi\big), \tag{7}$$

where $(u')^{-1}$ is the inverse of the derivative of the utility function u, φ is the density of P^* with respect to Q and the parameter λ is such that $E^*[X] = V_0$.

If the preferences are robust but the martingale measure P^* is still unique, then the optimisation problem can be solved by using a basic result from robust statistics. In their robust version of the Neyman–Pearson lemma, Huber and Strassen [13] had shown how to find a measure Q_0 which is *least favourable* in the class $\mathcal{Q}$ with respect to a given reference measure. In our financial context, Schied [18] proved that the solution to the optimisation problem is of the form (7) if φ denotes the density of the martingale measure P^* with respect to that measure Q_0 in the class $\mathcal{Q}$ which is least favourable with respect to P^*.

If on the other hand the martingale measure P^* is no longer unique but the preferences are of the classical form (3) for a single measure Q, then our optimisation problem reduces to a dual *projection problem*. More precisely, one has to project the measure Q onto the class $\mathcal{P}^*$ of martingale measures by minimising the functional

$$F(P^*|Q) = E_Q\big[f\big(\frac{dP^*}{dQ}\big)\big],$$

where f is a convex function obtained from the utility function u by a Legendre–Fenchel transformation. In the exponential case $u(x) = 1 - \exp(-\gamma x)$, the functional F is given by the relative entropy $H(P^*|Q)$, and we are back to a classical projection problem in probability theory. For more general utility functions u, the dual problem has been systematically studied, particularly by Kramkov and Schachermayer [14], Goll and Rüschendorf [11] and Bellini and Frittelli [2].

Let us now consider the general case where preferences are robust and the martingale measure is no longer unique. Now the problem consists in projecting the whole class $\mathcal{Q}$ onto the class $\mathcal{P}^*$. This amounts to finding a minimum of F, considered as a functional on the product set $\mathcal{P}^* \times \mathcal{Q}$. If one has identified a minimising couple (P_0^*, Q_0), then the solution to the optimisation problem is of the form (5) where φ is the density of Q_0 with respect to P_0^*. This robust version of a probabilistic projection problem, which is of intrinsic mathematical interest independently of the financial interpretation, has been studied by Gundel [12] and by Föllmer and Gundel [5].

It is, however, not always possible to find a solution to the dual problem within the class $\mathcal{P}^*$ of equivalent martingale measures. This difficulty may already appear if the preferences are of the classical form (3). In this case, Kramkov and Schachermayer [14] have shown how one may extend the class $\mathcal{P}^*$ in order to find a solution in some larger setting. For preferences of the form (5), new robust variants of the projection problem have recently been studied by Quenez [16], Schied and Wu [20] and Föllmer and Gundel [5]. Moreover, Schied [19] has just solved the optimisation problem for robust preferences in the general form (6).

Thus, we find that the maxim of humility when facing financial uncertainty is in fact a very rich source of new problems in probability theory.

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The Notion of Arbitrage and Free Lunch in Mathematical Finance

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Abstract We shall explain the concepts alluded to in the title in economic as well as in mathematical terms. These notions play a fundamental role in the modern theory of mathematical finance. We start by presenting the ideas in a very informal style and then gradually raise the level of mathematical formalisation.

1 Arbitrage

The notion of arbitrage is crucial in the modern theory of Finance. It is the cornerstone of the option pricing theory due to Black, Scholes [BS 73] and Merton for which they received the Nobel prize in Economics in 1997; Black died in 1995.

The underlying idea is best explained by telling a little joke: A finance professor and a normal person go on a walk and the normal person sees a €100 bill lying on the street. When the normal person wants to pick it up, the finance professor says: ‘Don’t try to do that! It is absolutely impossible that there is a €100 bill lying on the street. Indeed, if it were lying on the street, somebody else would already have picked it up before you’ (end of joke).

How about financial markets? There it is already much more reasonable to assume that there are no €100 bills lying around waiting to be picked up. We shall call such opportunities of picking up money that is ‘lying around’ arbitrage possibilities. Let us illustrate this with an easy example.

Consider the trading of \$ vs. € which takes place simultaneously at two exchanges, say in New York and Paris. Assume for simplicity that in New York the \$/€ rate is 1:1. Then it is quite obvious that in Paris the exchange rate (at the same moment of time) also is 1:1. Let us have a closer look why this is indeed the case. Suppose to the contrary that you can buy in Paris a \$ for €0.999. Then, indeed, the so-called ‘arbitrageurs’ (these are people with two telephones in their hands and three screens in front of them) would quickly act to buy \$ in Paris and simultaneously sell the same amount of \$ in New York, keeping the margin in their (or their bank's) pocket. Note that there is no normalising factor in front of the exchanged amount and the arbitrageur would try to do this on as large a scale as possible.

It is rather obvious that in the above described situation the market cannot be in equilibrium. A moment's reflection reveals that the market forces triggered by the arbitrageurs acting according to the above scheme will make the $ rise in Paris and fall in New York. The arbitrage possibility will only disappear when the two prices become equal. Of course 'equality' here is to be understood as an approximate identity where – even for arbitrageurs with very low (proportional) transaction costs – the above scheme is not profitable any more.

This brings us to a first – still informal and intuitive – definition of arbitrage: an arbitrage opportunity is the possibility to make a profit in a financial market *without risk* and *without net investment of capital*. The *principle of no arbitrage* states that a mathematical model of a financial market should not allow for arbitrage possibilities.

2 An Easy Model of a Financial Market

To apply this principle to less trivial cases, we consider a – still extremely simple – mathematical model of a financial market: there are two assets, called the bond and the stock. The bond is riskless, hence – by definition – we know what it is worth tomorrow. For (mainly notational) simplicity we neglect interest rates and assume that the price of a bond equals €1 today as well as tomorrow, i.e.

$$B_0 = B_1 = 1. \tag{1}$$

The more interesting feature of the model is the stock which is risky: we know its value today, say $S_0 = 1$, but we do not know its value tomorrow. We model this uncertainty stochastically by defining S_1 to be a random variable depending on the random element $\omega \in \Omega$. To keep things as simple as possible, we let Ω consist of two elements only, g for 'good' and b for 'bad', with probability $\mathbf{P}[g] = \mathbf{P}[b] = \frac{1}{2}$. We define $S_1(\omega)$ to equal 2 or $\frac{1}{2}$ according to whether

$$S_1(\omega) = \begin{cases} 2 & \text{for } \omega = g \\ \frac{1}{2} & \text{for } \omega = b. \end{cases} \tag{2}$$

Now we introduce a third financial instrument in our model, an *option on the stock* with strike price K: the buyer of the option has the right – but not the obligation – to buy one stock at time $t = 1$ at the predefined price K. To fix ideas let $K = 1$. A moment's reflexion reveals that the price C_1 of the option at time $t = 1$ (where C stands for *contingent* claim) equals

$$C_1 = (S_1 - K)_+, \tag{3}$$

i.e. in our simple example

$$C_1(\omega) = \begin{cases} 1 & \text{for } \omega = g \\ 0 & \text{for } \omega = b. \end{cases} \tag{4}$$

Hence we know the value of the option at time $t = 1$, *contingent on the value of the stock*. But what is the price of the option today?

At this stage the reader might consult the financial section of a newspaper or the web to see some 'life' examples on quoted option prices.

The classical approach, used by actuaries for centuries, is to price contingent claims by taking expectations, which leads to the value $C_0 := \mathbf{E}[C_1] = \frac{1}{2}$ in our example. Although this simple approach is very successful in many actuarial applications, it is not at all satisfactory in the present context. Indeed, the rationale behind taking the expected value as the price of a contingent claim is the following: in the long run the buyer of an option will neither gain nor lose on average. We rephrase this fact in a financial lingo: the performance of an investment in the option would on average equal the performance of the bond. However, a basic feature of finance is that an investment into a risky asset should, on average, yield a better performance than an investment in the bond (for the skeptical reader: at the least these two values should not necessarily coincide). In our 'toy example' we have chosen the numbers such that $\mathbf{E}[S_1] = 1.25 > 1 = \mathbf{E}[B_1]$, so that on average the stock performs better than the bond.

3 Pricing by No Arbitrage

A different approach to the pricing of the option goes like this: we can buy at time $t = 0$ a *portfolio* consisting of $\frac{2}{3}$ of stock and $-\frac{1}{3}$ of bond. The reader might be puzzled about the negative sign: investing a negative amount in a bond – 'going short' in financial lingo – means to borrow money.

One verifies that the value Π_1 of the portfolio at time $t = 1$ equals 1 or 0 depending on whether ω equals g or b. The portfolio 'replicates' the option, i.e.

$$C_1 \equiv \Pi_1. \tag{5}$$

We are confident that the reader now sees why we have chosen the above weights $\frac{2}{3}$ and $-\frac{1}{3}$: the mathematical complexity of determining these weights such that (5) holds true amounts to solving two linear equations in two variables.

The portfolio Π has a well-defined price at time $t = 0$, namely $\Pi_0 = \frac{2}{3}S_0 - \frac{1}{3}B_0 = \frac{1}{3}$. Now comes the 'pricing by no arbitrage' argument: equality (5) implies that we also must have

$$C_0 = \Pi_0 \tag{6}$$

whence $C_0 = \frac{1}{3}$. Indeed, suppose that (6) does not hold true; to fix ideas, suppose we have $C_0 = \frac{1}{2}$ as above. This would allow an arbitrage by buying ('going long in') the portfolio Π and simultaneously selling ('going short in') the option C. The difference $C_0 - \Pi_0 = \frac{1}{6}$ remains as arbitrage profit at time $t = 0$, while at time $t = 1$ the two positions cancel out *independently of whether the random element* ω *equals* g *or* b.

4 Variations of the Example

Although the preceding 'toy example' is extremely simple and, of course, far from reality, it contains the heart of the matter: the possibility of replicating a contingent claim, e.g. an option, by trading on the existing assets and applying the no arbitrage principle.

It is straightforward to generalise the example by passing from the time index set $\{0,1\}$ to an arbitrary finite discrete time set $\{0,\dots,T\}$ by considering T independent Bernoulli random variables. This binomial model is called the Cox–Ross–Rubinstein model in finance. It is not difficult – at least with the technology of stochastic calculus that is available today – to pass to the (properly normalised) limit as T tends to infinity, thus ending up with a stochastic process driven by Brownian motion. The so-called geometric Brownian motion with drift is the celebrated *Black–Scholes model*, which was proposed in 1965 by P. Samuelson. In fact, already in 1900 Bachelier used Brownian motion to prize options in his remarkable thesis 'Théorie de la spéculation' [B 00] (member of the jury and rapporteur: H. Poincaré).

In order to apply the above no arbitrage arguments to more complex models we still need one more crucial concept.

5 Martingale Measures

To explain this notion let us turn back to our 'toy example', where we have seen that the unique arbitrage-free price of our option equals $C_0 = \frac{1}{3}$. We also have seen that, by taking expectations, we obtained $\mathbf{E}[C_1] = \frac{1}{2}$ as the price of the option, which allowed for arbitrage possibilities. The economic rationale for this discrepancy was that the expected return of the stock was higher than that of the bond.

Now make the following thought experiment: suppose that the world is governed by a different probability than $\mathbf{P}$ that assigns different weights to g and b, such that under this new probability – let's call it $\mathbf{Q}$ – the expected return of the stock equals that of the bond. An elementary calculation reveals that the probability measure defined by $\mathbf{Q}[g] = \frac{1}{3}$ and $\mathbf{Q}[b] = \frac{2}{3}$ is the unique solution satisfying $\mathbf{E}_{\mathbf{Q}}[S_1] = S_0 = 1$. Speaking mathematically, the process S is a *martingale* under $\mathbf{Q}$, and $\mathbf{Q}$ a *martingale measure* for S.

Speaking again economically, it is not unreasonable to expect that in a world governed by $\mathbf{Q}$, the recipe of taking expected values should indeed give a price for the option that is compatible with the no arbitrage principle. A direct calculation reveals that in our 'toy example' this is indeed the case:

$$\mathbf{E_Q}[C_1] = \tfrac{1}{3}. \tag{7}$$

At this stage it is, of course, the reflex of every mathematician to ask: what precisely is going on behind this phenomenon?

6 The Fundamental Theorem of Asset Pricing

The basic message of this theorem is that – essentially – a model of a financial market is free of arbitrage if and only if there is a probability measure $\mathbf{Q}$, equivalent to the original $\mathbf{P}$ (i.e. $\mathbf{P}[A] = 0$ iff $\mathbf{Q}[A] = 0$), such that the stock price process is a martingale under $\mathbf{Q}$. In this case the recipe of taking expectations $\mathbf{E_Q}[\,.\,]$ in order to price contingent claims yields precisely the arbitrage-free pricing rules, where $\mathbf{Q}$ runs through all equivalent martingale measures. In particular, if $\mathbf{Q}$ is unique, $\mathbf{E_Q}[\,.\,]$ yields the unique arbitrage-free price, as in the 'toy example' above.

This theorem was proved by Harrison and Pliska [HP 81] in 1981 for the case where the underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is finite. In the same year Kreps [K 81] extended this theorem to a more general setting: for this extension the condition of no arbitrage turns out to be too narrow and has to be replaced by a stronger assumption.

7 No Free Lunch

We formalise the model of a financial market in continuous time $[0,T]$: the bond again is harmless and without loss of generality normalised, i.e. $B_t \equiv 1$ for $0 \leq t \leq T$. The stock price process $(S_t)_{0\leq t\leq T}$ is assumed to be an $\mathbb{R}^d$-valued stochastic process (we consider $d = 1$ for simplicity) defined over and adapted to a filtered stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$ such that the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the 'usual conditions' of right continuity and saturatedness.

A real-valued stochastic process $S = (S_t)_{0\leq t\leq T}$ is a function $S : \Omega \times [0,T] \to \mathbb{R}$ verifying some measurability condition: we call S progressively measurable if, for every $t \in [0,T]$, the restriction of S to $\Omega \times [0,t]$ is $\mathcal{F}_t \otimes \text{Borel}([0,t])$-measurable. The interpretation is that the behaviour of $(S_u)_{0\leq u\leq t}$ depends only on the information available at time t, which is modelled by the sigma-algebra $\mathcal{F}_t$.

The 'economic agents' are allowed to buy or sell arbitrary quantities of the stock during the time interval $[0,T]$. To model this activity mathematically, first consider so-called 'elementary' trading strategies: fix a mesh $0 = t_0 < t_1 < \ldots < t_n = T$, which we interpret as the (deterministic) instants of time when the agent rebalances her portfolio. To define the portfolio she has to determine the amounts $H_{t_{i-1}}$ of stock which she holds during the time intervals $]t_{i-1}, t_i]$. When she will make this decision, i.e. at time t_{i-1}, she will dispose of the information available at time $t-1$; hence it is natural to require $H_{t_{i-1}}$ to be an $\mathcal{F}_{t_{i-1}}$-measurable random variable. An elementary trading strategy therefore is defined as a function $H = H_t(\omega)$ defined on $\Omega \times T$ of the form

$$H_t(\omega) = \sum_{i=1}^{n} H_{t_{i-1}}(\omega)\mathbf{1}_{]t_{i-1},t_i]} \tag{8}$$

where each $H_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}}$-measurable. The interpretation is that at time t the agent holds $H_t(\omega)$ units of the stock in her portfolio.

For each such H we may define the stochastic integral X_T^H as the random variable

$$X_T^H = \sum_{i=1}^{n} H_{t_{i-1}}(S_{t_i} - S_{t_{i-1}}) =: \int_0^T H_t dS_t \tag{9}$$

The interpretation is as follows: applying the trading strategy H results in a (random) gain or loss $H_{t_{i-1}}(\omega)(S_{t_i}(\omega) - S_{t_{i-1}}(\omega))$ during the time interval $]t_{i-1}, t_i]$. The total gain or loss therefore is given by the above Riemann-type sum which may formally be written – for each fixed $\omega \in \Omega$ – as a Stieltjes integral dS_t over a step function.

This notation insinuates already that one should allow for more general trading strategies than just 'elementary' ones. Similarly as in ordinary calculus the step functions are only a technical gimmick on the way to a reasonable integration theory.

In order to pass to more general integrands H in (9) we need some more steps: firstly we have to suppose that S is a semi-martingale (see [RY 91] for a definition). This is, according to a theorem of Bichteler and Dellacherie, the maximal class of stochastic processes for which there is a reasonable integration theory. Fortunately, this assumption of S being a semi-martingale does not restrict the generality: indeed it was shown in [DS 94] that, whenever S fails to be a semi-martingale then – essentially (in a sense made precise in [DS 94]) – S admits arbitrage.

We can now proceed developing the integration theory. It is natural to replace the deterministic mesh $0 = t_0 \leq \ldots \leq t_n = T$ by an increasing sequence of *stopping times* $0 = \tau_0 \leq \ldots \leq \tau_n = T$. The corresponding class $\mathcal{H}$ of integrands (or trading strategies in the financial lingo) is called the class of *simple* integrands.

This first extension step (from elementary to simple integrands) is rather harmless and does not require any delicate passage to the limit as the integral is still defined by a finite sum as in (9).

In general, trading strategies are modelled by predictable, S-integrable processes $(H_t)_{0\leq t\leq T}$, where H_t describes the amount of the stock held at time t. For each H the random variable

$$X_T^H = \int_0^T H_t dS_t \tag{10}$$

then equals the accumulated gains or losses up to time T by following the trading strategy H. For technical reasons we have to restrict ourselves to 'admissible' trading strategies H: we require that there is some constant M (a 'credit line') such that, for all $0 \leq t \leq T$, the accumulated losses up to time t are less than M almost surely,

$$\int_0^t H_u dS_u > -M, \quad \text{a.s., for } 0 \leq t \leq T. \tag{11}$$

We now may formally define an arbitrage opportunity as an admissible trading strategy H s.t. the random variable X_T^H is non-negative a.s. and strictly positive with strictly positive probability.

We have mentioned above that the requirement that there are no arbitrage possibilities is too weak to imply the existence of an equivalent martingale measure in general. Kreps' idea was to allow for a passage to the limit:

Definition 7.1 ([K 81]). The process S admits a *free lunch*, if there is a random variable $f \in L_+^\infty(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{P}[f > 0] > 0$ and a net $(f_\alpha)_{\alpha \in I} = (g_\alpha - h_\alpha)_{\alpha \in I}$ such $g_\alpha = \int_0^T H_t^\alpha dS_t$, for some admissible trading strategy H^α, $h_\alpha \geq 0$ and $(f_\alpha)_{\alpha \in I}$ converges to f in the weak-star topology of $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$.

The economic idea behind this notion is the following: although f itself is not supposed to be of the form $\int_0^T H_t dS_t$, for some admissible H (this would be an arbitrage), we require that f can be approximated by f_α in a suitable topology. The interpretation of the random variables $h_\alpha \geq 0$ is that, in this approximation, people are allowed to 'throw money away'.

With these preparations we can deduce from D. Kreps' work the following version of the fundamental theorem of asset pricing.

Theorem 7.2. *A bounded process $S = (S_t)_{0 \leq t \leq T}$ admits no free lunch iff there is a probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ such that S is a martingale under $\mathbf{Q}$.*

Subsequent to Kreps' seminal paper many authors elaborated on improvements of this theorem (see, e.g. [DS 98] for an account on the literature). Typical questions are whether the weak-star topology (which is difficult to interpret economically) can be replaced by a finer topology or whether the net$(f_\alpha)_{\alpha \in I}$ can be replaced by a sequence $(f_n)_{n \in \mathbb{N}}$.

In [DS 94] we introduced the notion of a 'free lunch with vanishing risk' by replacing the weak-star topology in the above definition by the norm-topology of L^∞. One may then replace the net $(f_\alpha)_{\alpha \in I}$ by a sequence $(f_n)_{n=1}^\infty$; this notion allows for a clear-cut economic interpretation to which we tried to allude to by the term 'vanishing risk'.

To extend to the case of unbounded processes (which are important in the applications) one also needs some generalisations of the notions of a martingale, namely the notion of local martingales and sigma-martingales. The latter concept goes back to the work of Chou and Emery (see [DS 98]).

We now can state the version of the fundamental theorem of asset pricing as obtained in [DS 98].

Theorem 7.3. *A semi-martingale $S = (S_t)_{0 \leq t \leq T}$ admits no free lunch with vanishing risk iff there is a probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ such that S is a sigma-martingale under $\mathbf{Q}$.*

If S is bounded (resp. locally bounded) the term sigma-martingale may equivalently be replaced by the term martingale (resp. local martingale).

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Dynamic Financial Risk Management

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Abstract In this article, a global panorama of the interactions between Mathematics and Finance is presented. Some emphasis is put on several aspects of understanding risk management.

Introduction

For many years, financial markets have been part of people's everyday life. Information on the major securities can be found easily in the daily press or on the internet. The different actors of the economy look closely at some fundamental parameters, such as interest rates, currency rates, stock prices..., which may affect the management of their firms, especially after the deregulation process initiated in the United States in the 1970s. The accrued international dimension and efficiency of the communication means have extended this phenomenon to the entire world economy. Small investors savings and pensions are also exposed to the risks generated by variations in these factors.

Financial institutions are aware of the importance of managing these risks. They offer different solutions, which will be presented below. For instance, the following product can help small investors to deal with variations of the stock markets:

In 2005, bank XYZ offered the following product, which is a 5 years guaranteed capital fund:

- *0% of the yield of a basket of six indices when it is negative*
- *50% of the yield if lower than 50%*
- *25% + 100% of the yield if greater than 50%*

The basket is built with six financial indices from different international markets: FTSE 89, SMI, FTSE100, NIKKEI225, S&P500, Hang Seng.

More generally, in order to deal with variations in fundamental financial and economic factors, a large variety of financial instruments has been created. Simple derivative contracts (futures, options, swaps, etc.) or more exotic financial products (credit derivatives, catastrophe bonds, exotic options, etc.) are offered to private and institutional investors to transfer their financial risks to specialized financial institutions in exchange of a suitable compensation. The organized markets, on which these products are traded, are fully devoted to the management of (fundamental) financial risks. They can be seen as financial risk markets. As argued by Merton [25], the development of the financial risk industry would not have been possible without the development and support of theoretical tools. Mathematics has emerged as the leading discipline to address fundamental questions related to financial risk management. Mathematical finance, which applies the theory of probability and stochastic processes to finance, in particular Brownian motion and martingale theory, stochastic control and partial differential equations, is now a field of research on its own.

1 The Black–Scholes–Merton Paradigm of Zero-Risk

The trigger for the whole financial industry expansion has been surprisingly 'Brownian motion theory and Itô's stochastic calculus', first introduced in finance by Bachelier in his Thesis in 1900 [4], then used again by Black, Scholes and Merton in 1973 ([7] and [26]).

In particular, these authors have provided a method to price and hedge derivative instruments, which are financial contracts delivering a payoff $H(\omega)$, depending upon the scenario ω at a maturity date. The typical example is the (European) call option, which offers a protection in case of a large increase in the underlying asset price. More precisely, a European call provides its buyer with the right (and not the obligation) to purchase the risky asset at a pre-specified price (the exercise price) K at a pre-specified date T in the future (the maturity date). The potential gain at maturity can therefore be written as $(X_T - K)^+$, where X_T denotes the value of the underlying asset at maturity. Note that the product described in the introduction is in fact a combination of such options, based on a basket of six indices.

Black, Scholes and Merton have developed the completely new idea according to which it is possible for an option seller to deliver the contract at maturity without incurring any residual risk by using a dynamic trading strategy on the underlying asset. The stochastic arguments may discourage many people; it is however possible to reduce technical difficulties, and to develop arguments which are essentially probability-free, as first introduced by Foellmer [17].

1.1 A Dynamic Uncertain World

The uncertainty is modelled via a family Ω of scenarios ω, characterizing the possible trajectories of the asset prices in the future. Such paths are described as positive continuous functions ω with coordinates $X_t = \omega(t)$, such that the continuous quadratic variation exists: $[X]_t(\omega) = \lim_n \sum_{t_i \le t, t_i \in D_n} (X_{t_{i+1}} - X_{t_i})^2$ along the sequence of dyadic partitions D_n.

The pathwise version of stochastic calculus yields to Itô's formula

$$\begin{aligned} f(t, X_t(\omega)) &= f(0, x_0) + \int_0^t f'_x(s, X_s(\omega))\, dX_s(\omega) + \int_0^t f_t(s, X_s(\omega))\, ds \\ &\quad + \int_0^t \frac{1}{2} f''_{xx}(s, X_s(\omega)) d[X]_s(\omega) \end{aligned} \tag{1}$$

The last two integrals are well defined as Lebesgue–Stieltjes integrals, while the first exists as an Itô integral, defined as the limit of non-anticipating Riemann sums $\sum_{t_i \le t, t_i \in D_n} \delta_{t_i}(\omega)(X_{t_{i+1}} - X_{t_i})$, where $\delta_t = f'_x(t, X_t)$.

This formulation has a direct financial interpretation. More precisely, Itô's integral may be seen as the cumulative *gain process* of a trading strategy where δ_t is the number of the shares of the underlying asset held at time t and the increment in the Riemann sum is the price variation over the period. The non-anticipating assumption corresponds to the financial requirement that the investment decisions are based only on the observation of the past prices. At any time t, the residual wealth of the trader is invested in cash, yielding a rate (called short rate) r_t by time unit.

A natural condition is imposed on the strategy, forcing the associated portfolio to be *self-financing*. In other words, no money is added to or removed from the portfolio after its construction at time 0. This is expressed as:

$$dV_t = r_t(V_t - \delta_t.X_t)dt + \delta_t.dX_t = r_t V_t dt + \delta_t.(dX_t - r_t X_t\, dt),\ V_0 = z \tag{2}$$

1.2 Hedging Derivatives: Solving a Target Problem

We now come back to the problem of a trader having to pay at maturity T the amount $h(X_T(\omega))$ in the scenario ω (i.e. $h(X_T(\omega)) = (X_T(\omega) - K)^+$ for a European call option). The trader wants to hedge his position in the sense that this target payoff is replicated by the wealth generated by a self-financing portfolio in all scenarios at maturity. The 'miraculous' message in the world of Black and Scholes is that such a perfect hedge is possible and easily computable.

The additional assumption is that the quadratic variation is absolutely continuous $d[X]_t = \sigma(t, X_t) X_t^2 dt$.The (regular) strictly positive function $\sigma(t,x)$ is a key parameter in financial markets, called the local volatility.

Looking for the wealth at any time t as a function $f(t, X_t)$, we see that, given (1) and (2),

$$\begin{aligned} df(t, X_t) &= f'_t(t, X_t)dt + f'_x(t, X_t)dX_t + \frac{1}{2} f''_{xx}(t, X_t) X_t^2 \sigma^2(t, X_t)\, dt \\ &= f(t, X_t) r_t dt + \delta(t, X_t)\big(dX_t - X_t r_t\, dt\big) \end{aligned}$$

By identifying the dX_t terms (thanks to the assumption $\sigma(t,x) > 0$), we see that $\delta(t,X_t) = f'_x(t,X_t)$, and that f should be solution of the following partial differential equation, called pricing PDE:

$$f'_t(t,x) + \frac{1}{2} f''_{xx}(t,x)x^2\sigma^2(t,x) + f'_x(t,x)xr_t - f(t,x)r_t = 0,\; f(T,x) = h(x) \qquad (3)$$

The derivative price at time t_0 must be $f(t_0,x_0)$, if not, it is easy to generated profit without bearing any risk, which is an *arbitrage opportunity*. This is the *rule of the unique price*, which holds true in a liquid market.

Note that the volatility of the asset return is the only parameter of the underlying dynamics appearing in the PDE. The instantaneous return does not play any role, only the size of the fluctuations matters, not the direction. The fact that the option price does not depend on the market trend is a fundamental result. It could first seem surprising, as the main motivation of this financial product is to hedge the purchaser against underlying rises. But the possibility to replicate the option by a dynamic hedging strategy makes the framework risk-neutral and de-trend the market.

The PDE's fundamental solution $q(t,x,T,y)$ (with $h(x) = \delta_y(x)$) may be reinterpreted in terms of the Arrow–Debreu 'states prices' density, introduced in 1953 by this Nobel Prize winner, from a purely theoretical economical point of view and by completely different arguments.

The pricing rule of the derivative contract becomes: $f(t,x) = \int h(y)q(t,x,T,y)dy$, where q is the *pricing kernel*. When $\sigma(t,x) = \sigma_t$ is a deterministic function, the pricing kernel is the log-normal density, deduced from the Gaussian distribution by an explicit change of variable. The closed-form formula[1] for European call option price is the famous Black and Scholes formula.[2] The impact of this methodology was so important that Black, Scholes and Merton received the Nobel prize for economics in 1997.

2 From Market Risk to Model Risk

In practice, even in the simple Black–Scholes framework presented above (denoted by B&S in the following), a key problem is to calibrate the model parameters (in particular the volatility), the dynamics of which is given by the following stochastic differential equation (SDE):

[1] Obtaining closed-form formulae has a certain advantage, even if it is for very simple (over-simple) models, as they can be used as test functions for numerical procedure.

[2] In the Black–Scholes model with constant coefficients, the Call option price $C^{BS}(t,x,K,T)$ is given via the Gaussian cumulative function $\mathcal{N}(z) = \int_{\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$, and $\theta = T - t$,

$$\begin{cases} C^{BS}(t,x,t+\theta,K) = x\mathcal{N}\big(d_1(\theta,x/K)\big) - K e^{-r\theta}\mathcal{N}\big(d_0(\theta,x/K)\big) \\ d_0(\theta,x/K) = \frac{1}{\sigma\sqrt{\theta}}\mathrm{Log}\left(\frac{x}{Ke^{-r\theta}}\right) - \frac{1}{2}\sigma\sqrt{\theta}, \qquad d_1(\theta,x/K) = d_0(\theta,x/K) + \sigma\sqrt{\theta} \end{cases} \qquad (4)$$

Moreover $\Delta(t,x) = \partial_x C^{BS}(t,x,t+\theta,K) = \mathcal{N}\big(d_1(\theta,x/K)\big)$.

$$dX_t = X_t(\mu(t,X_t)dt + \sigma(t,X_t)dW_t),\ X_{t_0} = x_0$$

The Brownian motion W may be viewed as a standardized Gaussian noise with independent increments. The instantaneous return $\mu(t,X_t)$ is a trend parameter, that will not play any role in the derivative pricing as noticed in Sect. 1.2.

2.1 Statistical Vs. Implied Point of View

In the B&S model with constant parameter, the square of the volatility is the variance per unit time of the log return $R_t = \ln(X_t) - \ln(X_{t-h})$. If the only available information is the asset price historical data, the statistical estimator to be used is the empirical variance, computed over a more or less long time period $[0,N]$ as $\widehat{\sigma}^2 = \frac{1}{N-1}\sum_{i=0}^{N-1}(R_{t_i} - \overline{R_N})^2$ where $\overline{R_N} = \frac{1}{N}\sum_{i=0}^{N-1} R_{t_i}$. This estimator is called *historical volatility*.[3] Such an approach is very natural, especially in some new markets based on energy or on weather.

However, traders use this estimator with some reserve. They argue indeed that financial markets are not "statistically" stationary and that past is not enough to explain the future.

When possible, traders use the additional information given by the quoted option prices, $C^{obs}(T,K)$ for different maturities T and different strike prices K, and convert it into volatility parametrization. The *implied volatility* they obtain is implicitly defined as: $C^{obs}(T,K) = C^{BS}(t_0,x_0,T,K,\sigma^{imp})$. Moreover, the option hedge portfolio is easily computed by $\Delta^{imp} = \partial_x C^{BS}(t_0,x_0,T,K,\sigma^{imp})$. This strategy is used dynamically by defining the implied volatility and the associated Δ at any renegotiation dates. This specific use of the Black and Scholes formula based on the hedging strategy may explain its success.

2.2 The Option Real World is not Black and Scholes

This attractive methodology has rapidly appeared to be limited: observed implied volatilities are depending on the option parameters (time to maturity and exercise price). This implied volatility surface is in complete contradiction with the assumptions of the B&S model. In particular, the market quotes large movements, resulting in heavy tail distributions, higher than in the log-normal framework.

In order to take into account this empirical observation, the first idea has been to consider a model with local volatility $\sigma(t,x)$. In 1995, Dupire [13] related the implied volatility with the first and second partial derivatives of the option price by giving a clever formulation for the dual PDE (one dimensional in state variable).

[3] We do not write about the increasing research interest in statistical financial modelling as it is a subject in itself.

More precisely, when the market conditions are (t_0, x_0) and for the sake of simplicity the short rate is assumed to be null, then

$$C'_T(T,K) = \frac{1}{2}\sigma^2(T,K)\,K^2 C''_{KK}(T,K), \quad C(t_0,x_0) = (x_0 - K)^+. \tag{5}$$

where $C(T,K)$ is the price of the call option with parameters (T,K).

This formula is especially attractive as it gives a simple relation between 'quoted Call option prices'. The local volatility is computable from a continuum of *coherent* (without arbitrage) observed quoted prices as

$$\sigma^2(K,T) = 2\frac{C'_T(T,K)}{K^2 C''_{KK}(T,K)}\ .$$

2.3 Ill-Posed Inverse Problem for PDE

Unfortunately, the option market is typically limited to a relatively few different exercise prices and maturities; therefore, a naive interpolation yields to some irregularity and instability of the local volatility. The problem of determining local volatility can be viewed as a non-linear problem of function approximation from a finite data set. This data set is the value $C_{i,j}$ of the solution at (t_0, x_0) of PDE (3) with boundary conditions $h_{i,j}(x) = (x - K_{i,j})^+$ at maturity T_i. Setting the problem as a PDEs inverse problem yields to more robust calibration methods. These ideas appear for the first time in finance in 1995 [24], but the problem is not standard because of the strongly non-linear relationship existing between option prices and local volatility; moreover, the considered data set is related to a single initial condition. Prices adjustment is made through a least square minimization program, including a penalization term related to the local volatility regularity. Obtaining a numerical solution of this complex problem is extremely time consuming ([2], [3], [8], [9]).

2.4 Model Risk

In this local volatility problem, for instance, we may end up with several possible solutions depending on the selection criterion, in particular on the penalization that is used. Therefore, beyond the calibration problem there is a more fundamental issue related to the choice of a given model for the underlying dynamics. Choosing a particular model has an impact on the derivative prices and on the hedging strategy the trader obtains and therefore on the risk he undertakes.

The market risk itself appears to be not so important in the sense that it can be hedged (even partially) by using derivative instruments and replicating portfolios. Measuring the model risk is very difficult and barely taken into account by the agents in the market. This risk becomes all the more important since some financial risks cannot be hedged directly, as the volatility risk. To find the price of a derivative, the key argument of replication on which the B&S model is based, cannot be used any longer, as we will see in the next section.

3 Super-Replication and Risk Measures

The fundamental assumption of the B&S theory is that the options market may be entirely explained by underlying prices. In economic theory, it corresponds to a situation of market efficiency: a security price contains all the information relative to this particular security.

In an option world, the observed statistical memory of historical volatility leads naturally to consider stochastic volatility models. More precisely, let us assume that the volatility parameter σ of the asset price X has now its own specific risk, represented by a stochastic factor Y with $dY_t = \eta(t, X_t, Y_t)dt + \gamma(t, X_t, Y_t)\, d\tilde{W}_t$ where $\tilde{W}$ is a Brownian motion, which might be correlated with the Brownian motion driving X. What does it change? In fact, everything! Perfect replication by a portfolio is not possible any more; the notion of unique price does not exist any longer... But, such a situation, which appears to be non-standard, is often the general case. What kind of answer can we bring to such a problem?

3.1 Super-Replication and Robust Hedging

The option problem is still a target problem C_T, to be replicated by a portfolio $V_T(\pi,\delta) = \pi + \int_0^T \delta_s dX_s$ depending on market asset X. The investment decision (δ_t) is taken from available information at time t (in an adapted way).[4] Some other constraints (size, sign...) may also be imposed on δ. Let $\mathcal{V}_T$ be the set of all derivatives, that can be replicated at time T by an admissible portfolio. Their price at t_0 is given by the value of their replicating portfolio.

Super-replicating C_T is equivalent to find the smallest derivative $\widehat{C}_T \in \mathcal{V}_T$ which is greater than C_T in all scenarios. The super-replication price is the price of such a derivative. The $\widehat{C}_T$ replicating portfolio is the C_T robust hedging.

There are several ways to characterize the super-replicating portfolio:

1. Dynamic programming on level sets: This is the most direct (but least recent) approach. This method proposed by Soner, Mete and Touzi [29] has led the way for original works in geometry by giving a stochastic representation of a class of mean curvature-type geometric equations.
2. Duality: This second approach is based on a characterization of the 'orthogonal space' of $\mathcal{V}_T$ in terms of martingale measures in $\mathcal{Q}_T$, that is probability measures

[4] For the sake of simplicity, interest rates are assumed to be null and only one risky asset is traded on the market.

such that the expectation of the t-time value of non-negative gain portfolios is equal to 0. The super-replication price is then obtained as

$$\widehat{C}_0 = \sup_{\mathbf{Q} \in \mathcal{Q}_T} \mathbf{E}_{\mathbf{Q}}[C_T]. \tag{6}$$

Super-replication prices are often too expensive to be used in practice. However, they give an upper bound to the set of possible prices.

We will develop the dual approach in Sect. 3.3, when looking at risk measures. The super-replication price is indeed very similar to the notion of convex risk measures, where the worst case is also considered.

3.2 Martingale Measures

The idea of introducing a dual theory based on probability measures is first due to Bachelier [4] in 1900, even if it has been essentially developed by Harrison and Pliska [21]. The actual and most achieved form is due to Delbaen and Schachermayer [12] and to the very active group in theoretical mathematical finance.

More precisely, a *martingale measure*, $\mathbf{Q}$, is a probability measure characterized by: $\forall V_T \in \mathcal{V}_T$, $\mathbf{E}_{\mathbf{Q}}[V_T] = V_0$. When the set of admissible portfolios contains all simple strategies (the portfolio is renegotiated at discrete times, randomly chosen), the price of the fundamental asset (X_t) is a $\mathbf{Q}$-(local) martingale, in the sense that the best estimation under $\mathbf{Q}$ of X_{t+h} given the available information at time t is X_t. Therefore, the financial game is fair with respect to martingale measures.

3.2.1 Complete Market

When $\mathcal{V}_T$ contains all possible (path-dependent) derivatives, the market is said to be *complete*, and the set of martingale measures is reduced to a unique element $\mathbf{Q}$, often called risk-neutral probability. This is the case in the B&S framework. The security dynamics becomes $dX_t = X_t\,\sigma(t, X_t) dW_t^{\mathbf{Q}}$ where $W^{\mathbf{Q}}$ is a $\mathbf{Q}$-Brownian motion. This formalism is really efficient as it leads to the following *pricing rule*: $\widehat{C}_0 = \mathbf{E}_{\mathbf{Q}}(C_T)$. The computation of the option replicating portfolio is more complex. In the diffusion case, the price is function of risk factors and the replicating portfolio only depends on price partial derivatives.

3.2.2 Incomplete Market

A market is said to be *incomplete* in any other situation. The characterization of the set $\mathcal{Q}_T$ becomes all the more delicate since there are many different situations which may lead to market imperfections (non-tradable risks, transaction costs.. [11], [18],

[22], [23]). The abstract theory of super-replication (and more generally, of portfolio optimization under constraints) based on duality has been intensively developed; the super-replicating price process, described in Sect. 3.1 is proved to be a super-martingale with respect to any admissible martingale measure (see El Karoui and Quenez [15] or Kramkov [23]).

3.3 Risk Measures

After having considered the trader's problem of pricing and management of his risk (hedging or super-hedging), we now adopt the point of view of the risk manager himself, or more generally that of the regulator, who both have to deal with partially hedged positions most of the time. Indeed, when the perfect replication is not possible, or is too expensive, or when the model risk is too important, the trader has to measure his remaining market risk exposure. The traditional measure is the variance of the replicating error.

The regulator has a more global perspective and the risks he has to manage are those of the bank itself, i.e. the sum of all the remaining risk exposures of the different traders. The safety and perenniality of the bank depend on his measure of this global risk. Therefore, the variance appears to be over-simplistic, as it does not really focus on the downside part of the risk distribution. New criteria taking into account extreme events, have been imposed, transforming the risk management at both quantitative and qualitative levels (see [19]).

The Value at Risk (VaR) criterion, corresponding to the maximal level of losses acceptable with a given probability (usually 95%), has taken a considerable importance for several years. In 1995, the Basel Committee on Banking Supervision imposed market risk capital requirements for all financial institutions. These are based upon a daily VaR computation of the global risky portfolio. Such an assessment is essential from the operational point of view, as it impacts the provisions a bank has to hold to face market risks.

A huge debate has been however initiated by the academic world (ADEH[1]) on the VaR significance and consistency as a risk measure. In particular, its non-additive property, incompatible with the key idea of diversification, enables banks to play with subsidiary creations. This debate has received an important echo from the professional world, which is possibly planning to review this risk measure criterion. Sub-additive and coherent risk measures are an average estimation of losses with respect to a family of probability measures $\mathcal{Q}_\rho$ as $\rho(X) = \sup_{\mathbf{Q}\in\mathcal{Q}_\rho} \mathbf{E}_\mathbf{Q}(-X)$. This characterization has been extended to convex risk measures [19], by adding a penalization term on probability density, a typical example being the entropy.

The set of probability measures $\mathcal{Q}_\rho$ of the risk measure dual representation could describe different scenario anticipations or beliefs of the risk manager. The underlying framework is not necessarily dynamic, as it was previously when studying the dynamic hedging problem. But the set $\mathcal{Q}_\rho$ can be also closer to a dynamic market view in the sense that certain financial instruments have to be risk-free under these measures. In this latter case, the risk measure is very similar to a super-replication price (from the seller's point of view), previously described. Even if the motivations of the risk manager and of the trader are a priori completely different, they end up with the same assessment of their risk, as soon as no perfect hedge is possible and some unhedgeable risks remain.

4 A New Research Field: Securitization

As described by Cummins in [10], '*securitization* is one of the most important innovations of modern finance. The securitization process involves the isolation of a pool of assets or rights to a set of cash flows and the repackaging of the assets or cash flows into securities that are traded in capital markets. The trading of cash flow streams enables the different parties to the contract to manage and diversify their risk, to take advantages of arbitrage opportunities or to invest in new classes of risk that enhance market efficiency'.

4.1 New Products

More precisely, the first securitization of bank loans took place in the 1970s in the United States. Several banks pooled their mortgages together, repackaged them and sold them on financial markets, creating a new type of securities, called Mortgage-Backed Securities. Since then, this securitization process has been generalized to other banking activities. Some complex structures have been recently issued on the credit risk market: the famous Collateralized Debt Obligations (CDOs), consisting of the repackaging of cash flows subject to credit risk, i.e. risk of default of the counterpart. By this securitization process, the individual risk is reduced through diversification. This creates however a new risk related to the correlation and joint defaults of the different counterparts of the banks.

From a modelling point of view, risks have been represented in a continuous way so far. Now, modelling potential losses dynamically and allowing assets to default require the introduction of some processes with jumps. This possibility of default makes the financial market incomplete, the default being typically unhedgeable. Using all the theoretical framework developed previously in a continuous framework, new approaches of the dynamic hedging problem have to be adopted to take into account the new discontinuities.

4.2 New Perspectives at the Interface Between Finance and Insurance

The past decade has seen the emergence of a range of financial instruments depending on risks traditionally considered to be within the remit of the insurance sector. Examples are weather and catastrophe claims, contingent on the occurrence of certain weather or catastrophic events. The development of instruments at the interface of insurance and finance raises new questions, not only about their classification but also about their design, pricing and management. The pricing issue is particularly intriguing as it questions the very logic of such contracts. Indeed, standard principles for derivatives pricing based on replication do not apply any more because of the special nature of the underlying risk. On the other hand, the question of the product design, unusual in finance, is raised since the logic behind these products is closer to that of an insurance policy (see for instance Barrieu and El Karoui [5]).

4.3 Problems Related to the Dimension

A key problem associated with the development of these products is the repackaging of the cash flows themselves just as the price. The high dimension of the problem makes these questions even harder. Several techniques have been recently developed to tackle this issue.

Identifying the different risk directions and the impact of estimation methods (for instance, when computing the VaR, the number of observations is less than that of risk factors...) is crucial not only to study a bank portfolio VaR, but also to hedge derivatives depending on a large number of underlying assets [16]. Random matrix theory or other asymptotic tools may bring some new ideas on this question.

Finally, by presenting the most important tools of the financial risk industry, we have voluntarily left apart anything on financial asset statistical modelling, which may be the subject of a whole paper on its own. It is clear that the VaR criterion, the market imperfections are highly dependent on an accurate analysis of the real and historical worlds. Intense and very innovating research is now developed on related topics (high-frequency data, ARCH and GARCH processes, Lévy processes, processes with long memory, random cascades).

Conclusion

As a conclusion, applied mathematicians have been highly questioned by problems coming from the financial risk industry. This is a very active, rapidly evolving world in which theoretical thoughts often have immediate and practical impacts and applications. Moreover, practical constraints generate new theoretical problems. This presentation is far from being an exhaustive view of the financial problems. Many exciting problems, from both theoretical and practical points of view, have not been presented.

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Stochastic Clock and Financial Markets

H. Geman

Abstract Brownian motion played a central role throughout the twentieth century in probability theory. The same statement is even truer in finance, with the introduction in 1900 by the French mathematician Louis Bachelier of an arithmetic Brownian motion (or a version of it) to represent stock price dynamics. This process was 'pragmatically' transformed by Samuelson in ([48, 49]; see also [50]) into a geometric Brownian motion ensuring the positivity of stock prices.

More recently the elegant martingale property under an equivalent probability measure derived from the No Arbitrage assumption combined with Monroe's theorem on the representation of semi-martingales have led to write asset prices as time-changed Brownian motion. Independently, Clark [14] had the original idea of writing cotton Future prices as subordinated processes, with Brownian motion as the driving process. Over the last few years, time changes have been used to account for different speeds in market activity in relation to news arrival as the stochastic clock goes faster during periods of intense trading. They have also allowed us to uncover new classes of processes in asset price modelling.

1 Introduction

The twentieth century started with the pioneer dissertation of Bachelier [3] and the introduction of Brownian motion for stock price modelling. It also ended with Brownian motion as a central element in the representation of the dynamics of such assets as bonds, commodities or stocks. In the mean time, the reasonable assumption of the non-existence of arbitrage opportunities in financial markets led to the First Fundamental Theorem of Asset Pricing. From the representation of discounted asset prices as martingales under an equivalent martingale measure, the semi-martingale property for asset prices under the real probability measure, was derived then in turn the expression of these prices as time changed Brownian motion.

Since the seminal papers by Black and Scholes [9] and Merton [43] on the pricing of options, the theory of No Arbitrage has played a central role in finance. It is in fact amazing how much can be deduced from the simple economic assumption that it is not possible in a financial market to make profits with zero investment and without bearing any risk. Unsurprisingly, practitioners in various sectors of the economy are prepared to subscribe to this assumption, hence to harvest the flurry of results derived from it. Pioneering work on the relation between No Arbitrage arguments and martingale theory was conducted in the late seventies and early eighties by Harrison–Kreps–Pliska: Harrison and Kreps [33] introduce in a discrete-time setting the notion of *equivalent martingale measure*. Harrison and Pliska [34] examine the particular case of complete markets and establish the *unicity* of the equivalent martingale measure. A vast amount of research grew out of these remarkable results: Dalang, Morton and Willinger [16] extend the discrete-time results to a general probability space Ω. Delbaen and Schachermayer [18] choose general semi-martingales for the price processes of primitive assets and establish the
First Fundamental Theorem of Asset Pricing:

The market model is arbitrage free if and only if there exists a probability measure Q *equivalent to P (and often called equivalent martingale measure) with respect to which the discounted prices of primitive securities are martingales.*

We consider the classical setting of a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ whose filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ represents the flow of information accruing to the agents in the economy, that is right-continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$; we are essentially considering in this paper a finite horizon T. The security market consists of (n+1) primitive securities: $(S_i(t))_{0 \leq t \leq T}, i = 1 \ldots n$ denotes the price process of the n stocks and S_0 is the money-market account that grows at a rate r supposed to be constant in the Black–Scholes–Merton and Harrison–Kreps–Pliska setting. At the start, the process S is only assumed to be locally bounded, a fairly general assumption which covers in particular the case of continuous price processes. We assume that the process S is a semi-martingale, adapted to the filtration $(\mathcal{F}_t)$ and satisfying the condition of being right-continuous and limited to the left. The semi-martingale property has to prevail for the process S in an arbitrage free market: by the First Fundamental Theorem of Asset Pricing mentioned above, the discounted stock price process is a martingale under an equivalent martingale measure; hence the stock price process has to be a semi-martingale under the initial probability measure P.

A self-financing portfolio Π is defined as a pair (x, H), where the constant x is the initial value of the portfolio and $H = \left(H^i\right)_{0 \leq i \leq n}$ is a predictable S-integrable process specifying the amount of each asset in the portfolio.

The value process of such a portfolio Π at time t is given by

$$V(t) = x_0 + \int_0^t H_u \cdot dS_u \quad 0 \leq t \leq T$$

In order to rule out strategies generating arbitrage opportunities by going through times where the portfolio value is very negative, Harrison and Pliska [34] define a predictable, S-integrable process H as admissible if there exists a positive constant c such that

$$(H.S)_t = \int_0^t H_u \cdot dS_u \geq -c \quad \text{for } 0 \leq t \leq T$$

This condition has been required in all the subsequent literature; it also has the merit to be consistent with the reality of financial markets since margin calls imply that losses are bounded.

In the particular case of a discrete-time representation, each R^d-valued process $(H_t)_{t=1}^T$ which is predictable (i.e. each H_t is $\mathcal{F}_{(t-1)}$ measurable) is S-integrable and the stochastic integral $(H \cdot S)_T$ reduces to a sum

$$(H.S)_T = \int_0^T H_u \cdot dS_u = \sum_{u=1}^T H_u \cdot (\Delta S_u) = \sum_{u=1}^T H_u \cdot (S_u - S_{u-1})$$

where $H_u . \Delta S_u$ denotes the inner product of the vectors H_u and $\Delta S_u = S_u - S_{u-1}$ which belong to R^d. Of course, such a trading strategy H is admissible if the underlying probability space Ω is finite.

We define a contingent claim as an element of $L^\infty(\Omega, \mathcal{F}, P)$. A contingent claim C is said to be attainable if there exists a self-financing trading strategy H whose terminal value at date T is equal to C. Assuming momentarily zero interest rates for simplicity, we denote by A_0 the subspace of $L^\infty(\Omega, \mathcal{F}, P)$ formed by the random variables $(H \cdot S)_T$ – representing the value at time T of attainable contingent claims – and by J the linear space spanned by A_0 and the constant 1. The no-arbitrage assumption implies that the set J and the positive orthant with the origin deleted, denoted as K, have an empty intersection. Hence, from the Hahn–Banach theorem, there exists (at least) a hyperplane G containing J and such that $G \cap K = \phi$.

We can then define the linear functional χ by $\chi/G = 0$ and $\chi(1) = 1$. This linear functional χ may be identified with a probability measure Q on $\mathcal{F}$ by $\chi = \frac{dQ}{dP}$ and χ is strictly positive if and only if the probability measure Q is equivalent to P. In addition, χ vanishes on A_0 if and only if S is a martingale under Q and this provides a brief proof of the First Fundamental Theorem of Asset Pricing (the other implication being simple to demonstrate). The proof is extended to non-zero (constant) interest rates in a non-elementary manner (see Artzner and Delbaen [2]) and to stochastic interest rates in Geman [26].

2 Time Changes in Mathematics

2.1 Time Changes: The Origins

The presence of time changes in probability theory can be traced back to the French mathematician Doeblin who, in 1940, studies real-valued diffusions and writes the 'fundamental martingales' attached to a diffusion as time-changed Brownian motion. Volkonski [51] uses time changes to study Markov processes. A considerable contribution to the subject was brought by Itô and McKean [36] (see also Feller [22]) who show that time changes allow to replace the study of diffusions by the study of

Brownian motion. McKean [42], in his beautifully entitled paper 'Scale and Clock', revisits space and time transformations and how they allow to reduce the study of complex semi-martingales to that of more familiar processes. In the framework of finance and in continuity with our previous section on the martingale representation of discounted stock prices, we need to mention the theorem by Dubins and Schwarz [19] and Dambis [17]:

Any continuous martingale is a time-changed Brownian motion.

The time-change which transforms the process S to this new scale is the quadratic variation, which is the limit of the sum of squared increments when the time step goes to zero. For standard Brownian motion, the quadratic variation over any interval is equal to the length of the interval. If the correct limiting procedure is applied, then the sum of squared increments of a continuous martingale converges to the quadratic variation. This *quadratic variation* has recently become in finance the subject of great attention with the development of such instruments as variance swaps and options on volatility index (see Carr–Geman–Madan–Yor [13]).

Continuing the review of the major mathematical results on time changes, we need to mention the theorem by Lamperti [39] which establishes that the exponential of a Lévy process is a time-changed self-similar Markov process. Obviously, a Brownian motion with drift is a particular case of a Lévy process. Williams [52] shows that the exponential of Brownian motion with drift is a time-changed Bessel process. It is on this representation of a *geometric Brownian motion as a time-changed squared Bessel process* that Geman and Yor [30] build their exact valuation of Asian options in the Black–Scholes setting; in contrast to geometric Brownian motion, the class of squared Bessel processes is stable by additivity and this property is obviously quite valuable for the pricing of contingent claims related to the arithmetic average of a stock or commodity price.

Thirteen years after the Dubins–Schwarz theorem, Monroe [45] extends the result to semi-martingales and establishes that

Any semi-martingale can be written as a time-changed Brownian motion.

More formally it is required that there exists a filtration $(\mathcal{G}_u)$ with respect to which the Brownian motion W(u) is adapted and that T(t) is an increasing process of stopping times adapted to this filtration $(\mathcal{G}_u)$.

2.2 Market Activity and Transaction Clock

The first idea to use such a clock for analyzing financial processes was introduced by Clark [14]. Clark was analyzing cotton Futures price data and in order to address the non-normality of observed returns, wrote the price process as a *subordinated process*

$$S(t) = W(X(t))$$

where he conjectured that:

- The process W had to be Brownian motion.
- The economic interpretation of the subordinator X was the cumulative volume of traded contracts.

Note that subordination was introduced in harmonic analysis (and not in probability theory) by Bochner in [10] and that subordinators are restrictive time changes as they are increasing processes with independent increments.

Ané and Geman [1] analyse a high frequency database of S&P future contracts over the period 1993–1997. They exhibit the increasing deviations from normality of realised returns over shorter time intervals.

Descriptive statistics of S&P 500 future prices at various time resolutions over the period 1993–1997

	Mean	Variance	Skewness	Kurtosis
1 min	1,77857E-6	8,21128E-08	1,109329038	58,59028989
15 min	2,50594E-05	1,1629E-06	−0,443443082	13,85515387
30 min	4,75712E-05	1,95136E-06	−0,092978342	6,653238051
Hour by hour	8,76093E-05	3,92529E-06	−0,135290879	5,97606376

Revisiting Clark's brilliant conjecture[1], they demonstrate that the structure of semi-martingales necessarily prevails for stock prices S by bringing together the No Arbitrage assumption and Monroe's theorem to establish that any stock price may be written as

$$S(t) = W(T(t))$$

where W is Brownian motion and T is an almost surely increasing process.

They show, in a general non-parametric setting, that in order to recover a quasi perfect normality of returns, the transaction clock is better represented by the number of trades than by the volume. In [37], Jones, Kaul and Lipton had exhibited that conditional on the number of trades, the volume was hardly an explanatory factor for the volatility.

Moreover Ané and Geman [1] show that, under the assumption of independence of W and T, the above expression of ln S(t) leads to the representation of the return as a mixture of normal distributions, in line with the empirical evidence exhibited by Harris [32].

Conducting the analysis of varying market activity in its most obvious form, a vast body of academic literature has examined trading/non-trading time effects on the risk and return characteristics of various securities; non-trading time refers to those periods in which the principal market where a security is traded is closed (and the transaction clock goes at a very low pace). Trading time refers to the period in which a security is openly traded in a central market (e.g. NYSE, CBOT) or an active over-the-counter market. These studies (Dyl and Maberly, [20]) first focused on differing returns/variances between weekdays and weekends. Subsequent studies (Geman and Schneeweis, [29]) also tested for intertemporal changes in asset risk as measured by returns variance of overnight and daytime periods as well as intraday time intervals. Results in both the cash (French and Roll, [23]) and the futures markets (Cornell [15]) indicate greater return variance during trading time than during

[1] I am grateful to Joe Horowitz for bringing to my attention Monroe's paper during Summer 1997.

non-trading time. Geman and Schneeweis [29] argue that 'the non-stationarity in asset return variance should be discussed in the context of calendar time and transaction time hypotheses'. French and Roll [23] conduct an empirical analysis of the impact of information on the difference between trading and non-trading time stock return variance. They conclude that information accumulates more slowly when the NYSE and AMEX are closed, resulting in higher volatility in these markets after weekends and holidays.

French, Schwert and Stambaugh [24], Samuelson [50] show that expected returns are higher for more volatile stocks since investors are rewarded for taking more risk. Hence, the validity of the semi-martingale model discussed in the previous section for stock prices: the sum of a martingale and a trend process, which is unknown but assumed to be fairly smooth, continuous and locally of finite variation.

2.3 Stochastic Volatility and Information Arrival

Financial markets go through hectic and calm periods. In hectic markets, the fluctuations in prices are large. In calm markets, price fluctuations tend to be moderate. The simplest representation of the size of fluctuations is volatility, the central quantity in financial markets. Financial time series exhibit, among other features, the property that periods of high volatility – or large fluctuations of the stock or commodity price-tend to cluster as shown in Figs. 1 and 2.

A considerable amount of literature, including Mandelbrot and Taylor [41], Clark [14], Karpoff [38], Harris [32], Richardson and Smith [47] have suggested to link asset return volatility to the flow of information arrival. This flow is not uniform through time and not always directly observable. Its most obvious components include quote arrivals, dividend announcements, macro-economic data release or markets closures. Figure 3 shows the dramatic effect on the share price of the oil company Royal Dutch Shell produced by the announcement in January 2004 of a large downward adjustment in the estimation of oil reserves owned by the company. At the same time, oil prices were sharply increasing under the combined effect of growth in demand and production uncertainties in major producing countries.

Geman and Yor [30] propose to model a non-constant volatility by introducing a clock which measures financial time: the clock runs fast if trading is brisk and runs slowly if trading is light. We can observe the property in a deterministic setting: by self-similarity of Brownian motion, an increase in the scale parameter σ may be interpreted as an increase in speed

$$(\sigma W(t), t \geq 0) \overset{\text{law}}{=} W\left(\sigma^2 t\right) \; t \geq 0, \quad \text{for any } \sigma > 0$$

Hence, volatility appears as closely related to time change: doubling the volatility σ will speed up the financial time by a factor four. Bick [8] revisits portfolio insurance strategies in the context of stochastic volatility. Instead of facing an unwanted outcome of the portfolio insurance strategy at the horizon H, he suggests to roll the strategy at the first time τ_b such that

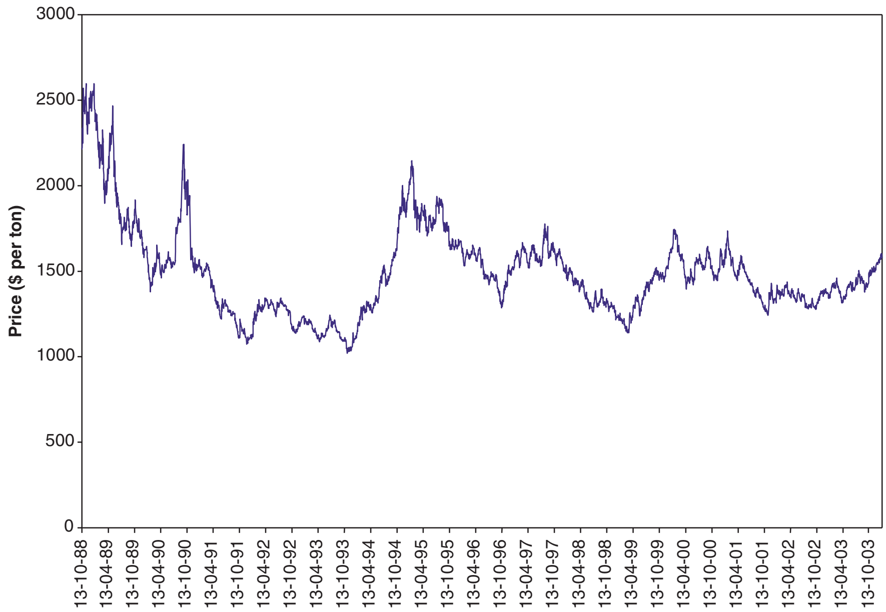

Fig. 1 Aluminum nearby Future prices on the London Metal Exchange from October 1988 to December 2003

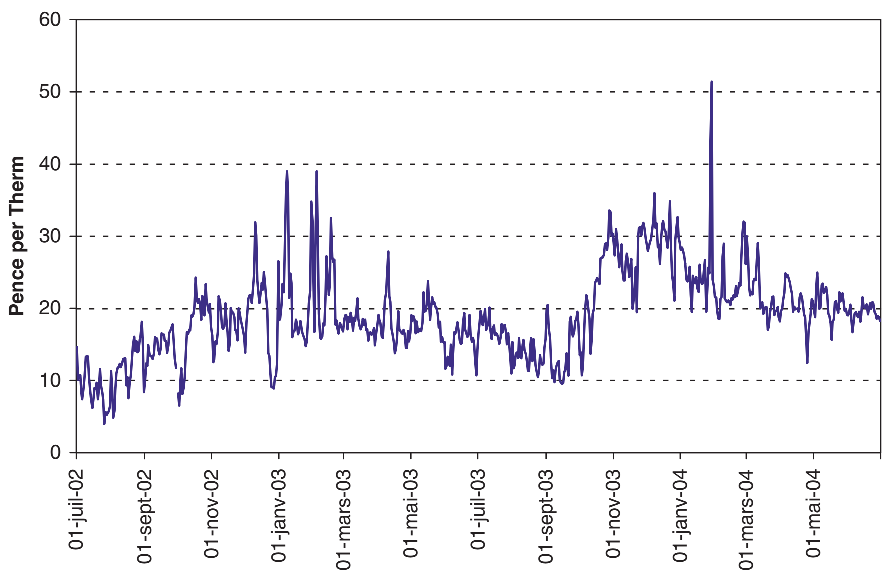

Fig. 2 UK National Balancing Point gas price

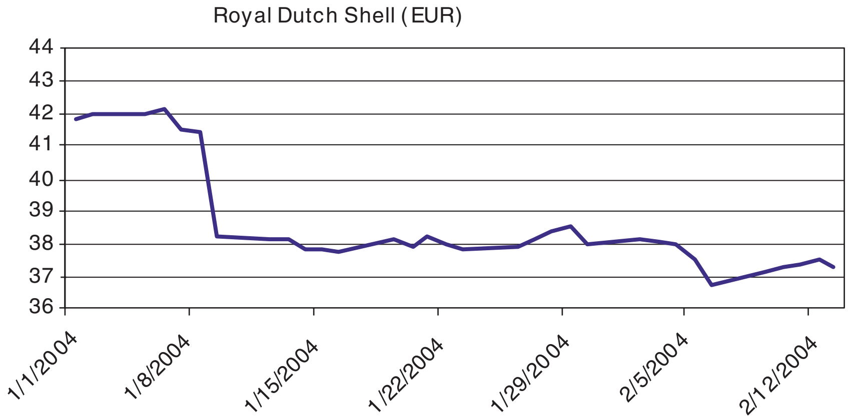

Fig. 3 Royal Dutch Shell price over the period January 1 to February 12, 2004

$$\int_0^{\tau_b} \sigma^2(s)\,ds = b \tag{1}$$

where b (b > 0) is the volatility parameter chosen at inception of the portfolio strategy at date 0. Assuming that σ (t) is continuous and $\mathcal{F}_t$- adapted, it is easy to show that the stopping time

$$\tau_b = \inf\left\{ u \geq 0 :\ \int_0^u \sigma^2(s)\,ds = b \right\} \tag{2}$$

is the first instant at which the option replication is correct, and hence, the portfolio value is equal to the desired target.

Geman and Yor [30] look at the distribution of the variable τ_b in the Hull and White [35] model of stochastic volatility where both the stock price and variance are driven by a geometric Brownian motion

$$\begin{cases} \dfrac{dS(t)}{S(t)} = \mu_1\,dt + \sigma(t)\,dW^1(t) \\ \dfrac{dy(t)}{y(t)} = \mu_2\,dt + \eta(t)\,dW^2(t) \end{cases}$$

where $d\langle W^1, W^2\rangle_t = \rho\,dt$ and $y\,(t) = [\sigma\,(t)]^{\,2}$

The squared volatility following a geometric Brownian motion can be written as a squared Bessel process through Lamperti's [39] theorem.

Hence the volatility itself may be written as

$$\sigma\,(t) = R_{\sigma_0}^{(v)}\left(4\int_0^t \sigma^2\,(u)\,du\right)$$

where $\left(R_{\sigma_0}^{(v)}\,(u)\right)_{u\geq 0}$ is a Bessel process starting at σ_0, with index

$$v = \frac{2\mu_2}{\eta^2} - 1$$

In order to identify the stopping time τ_b, we need to invert Eq. (1) and this leads to

$$\tau_b = \int_0^b \frac{du}{\left[R_{\sigma_0}^{(v)}(u)\right]^2}$$

The probability density f_b of τ_b does not have a simple expression but its Laplace transform has an explicit expression (see Yor [53]):

$$\int_0^{+\infty} f_b(x) e^{-\lambda x} dx = \frac{1}{\Gamma(k)} \int_0^1 \exp\left(\frac{-2u\sigma_0^2}{b\eta^2}\right)\left(\frac{2u\sigma_0^2}{b\eta^2}\right)^k (1-u)^{\mu-k} du$$

where $\mu = \left(\frac{8\lambda}{\eta^2} + v^2\right)^{\frac{1}{2}}$ and $k = \frac{\mu - v}{2}$.

Eydeland and Geman [21] propose an inversion of this Laplace transform. By linearity, the same method can apply to obtain the expectation of τ_b, i.e. the average time at which replication will be achieved and the roll of the portfolio insurance strategy take place. Note that the same type of time change technique can apply to other models of stochastic volatility and the option trader can compare the different answers obtained for the distributions of the stopping time of exact replication.

2.4 Stochastic Time and Jump Processes

Geman, Madan and Yor [28] – hereafter GMY – argue that asset price processes arising from market clearing conditions should be modelled as pure jump processes, with no continuous martingale component. Since continuity and normality can always be obtained after a time change, they study various examples of time changes and show that in all cases, they are related to measures of economic activity. For the most general class of processes, the time change is a size-weighted sum of order arrivals.

The possibility of discontinuities or jumps in asset prices has a long history in the economics literature. Merton [44] considered the addition of a jump component to the classical geometric Brownian motion model for the pricing of options on stocks. Bates [7], Bakshi, Cao and Chen [4] propose models that contain a diffusion component in addition to a low or finite activity jump part. The diffusion component accounts for high activity in price fluctuations while the jump component is used to account for rare and extreme movements. By contrast, GMY account for the small, high activity and rare large moves of the price process in a unified and connected manner: all motion occurs via jumps. High activity is accounted for by a large (in fact infinite) number of small jumps. The various jump sizes are analytically connected by the requirement that smaller jumps occur at a higher rate than larger jumps. In this family of Lévy processes, the property of an infinite number of small moves is shared with the diffusion-based models, with the additionally attractive feature that the sum of absolute changes in price is finite while for diffusions, this quantity is infinite (for diffusions, the price changes must be squared before they sum to a finite value). This makes possible the design and pricing of contracts such as 'droptions' based on the instantaneous upward, downward or total variability (positive, negative or absolute price jump size) of underlying asset prices, in addition to the more traditional contracts with payoffs that are functionally related to the level of the underlying price. These processes include the α-stable processes (for $\alpha < 1$) that were studied by Mandelbrot [40].

The empirical literature that has related price changes to measure of activity (see for instance Karpoff [38], Gallant, Rossi and Tauchen [25] and Jones, Kaul and Lipton [37]) has considered as relevant measures of activity either the number of trades or the volume. Geman, Madan and Yor [28] argue that time changes must be processes with jumps that are locally uncertain, since they are related to demand and supply shocks in the market.

Writing S(t) = W (T(t)) we see that the continuity of (S(t)) is equivalent to the continuity of (T(t)). If the time change is continuous, it must essentially be a stochastic integral with respect to another Brownian motion (see Revuz and Yor [46]): denoting T(t) this time change, it must satisfy an equation of the type

$$dT(t) = a(t)\,dt + b(t)\,dB(t)$$

where (B(t)) is a Brownian motion. For the time change to be an almost surely increasing process, the diffusion component b(t) must be zero. Then the time change would be locally deterministic, which is in contradiction with its fundamental interpretation in terms of supply and demand order flow.

The equation

$$S(t) = W(T(t))$$

implies that the study of price processes for market economies may be reduced to the study of time changes for Brownian motion. We can note that this is a powerful reduced-form representation of a complex phenomenon involving multidimensional considerations – those of modelling supply, demand and their interaction through market clearing – to a single entity: the correct unit of time for the economy with respect to which we have a Brownian motion.

Hence, the investigations may focus on theoretically identifying and interpreting T(t) from a knowledge of the process S(t) through historical data. GMY define a process as exhibiting a high level of activity if there is no interval of time in which prices are constant throughout the time interval.

An important structural property of Lévy densities attached to stock prices is that of monotonicity. One expects that jumps of larger sizes have lower arrival rates than jumps of smaller sizes. This property amounts to asserting for differentiable densities that the derivative is negative for positive jump sizes and positive for negative jump sizes. We may want to go further in that direction and introduce the property of complete monotonicity for the density. If we focus our attention on the density corresponding to positive jumps (this does not mean that we assume symmetry of the Lévy density), a completely monotone Lévy density on R^+ will be decreasing and convex, its derivative increasing and concave and so on. Structural restrictions of this sort are useful in limiting the modelling set, given the wide class of choices that are otherwise available to model the Lévy density, which basically is any positive function that integrates the minimum of x^2 and 1. Complete monotonicity has the interesting property of linking analytically the arrival rate of large jumps to that of small ones by requiring the latter to be larger than the former. The presence of such a feature makes it possible to learn about larger jumps from observing smaller ones. In this regard, we note that the Merton [44] jump diffusion model is not completely monotone as the normal density shifts from being a concave function near zero to a convex function near infinity. On the other hand, the exponentially distributed jump size is the foundation for all completely monotone Lévy densities (accordingly, they have been largely used in insurance to model losses attached to weather events).

2.5 Stable Processes as Time-Changed Brownian Motion

For an increasing stable process of index $\alpha < 1$, the Lévy measure is

$$v(dx) = \frac{1}{x^{\alpha+1}}\,dx \quad \text{for } x > 0$$

The difference X(t) of two independent copies of such a process is the symmetric stable process of index α with characteristic function

$$E\left[\exp\left(iuX(t)\right)\right] = \exp\left(-tc\,|u|^{\alpha}\right)$$

for a positive constant c.

If we compute now the characteristic function of an independent Brownian motion evaluated at an independent increasing stable process of index α, we obtain

$$E\left[\exp\left(iuW(T(t))\right)\right] = E\left[\exp\left(-u^2 T(t)/2\right)\right] = \exp\left(-t\,(c/2)\,|u|^{2\alpha}\right)$$

or a symmetric stable process of index 2α.

It follows from this observation that the difference of two increasing stable α processes for $\alpha < 1$, is Brownian motion evaluated at an increasing stable $\alpha/2$ process.

2.6 The Normal Inverse Gaussian Process

Barndorff-Nielsen [5] proposed the normal inverse Gaussian (NIG) distribution as a possible model for the stock price. This process may also be represented as a time-changed Brownian motion, where the time change T(t) is the first passage time of another independent Brownian motion with drift to the level t. The time change is therefore an *inverse Gaussian process*, and as one evaluates a Brownian motion at this time, this suggests the nomenclature of a NIG process.

We note that the inverse Gaussian process is a homogeneous Lévy process that is in fact a stable process of index $\alpha = \frac{1}{2}$. We observed above that if $2\ \alpha < 1$, time changing Brownian motion with such a process leads to the symmetric stable process of index $\alpha < 1$. For $\alpha = \frac{1}{2}$, we show below that the process is of infinite variation. In general, for W(T(t)) to be a process of bounded variation, we must have that $\int (1 \wedge |x|)\, \tilde{v}\,(dx) < \infty$ where $\tilde{v}$ is the Lévy measure of the time-changed Brownian motion.

Returning to the expression of the NIG process, it is defined as

$$X_{NIG}\,(t; \sigma, v, \theta) = \theta T_t^v + \sigma W\left(T_t^V\right)$$

where, for any positive t, T_t^v is the first time a Brownian motion with drift v reaches the positive level t. The density of T_t^v is inverse Gaussian and its Laplace transform has the simple expression

$$E\left[\exp\left(-\lambda T_t^v\right)\right] = \exp\left[-t\left(\sqrt{2\lambda + v^2} - v\right)\right]$$

This leads in turn to a fairly simple expression of the characteristic function of the NIG process in terms of the three parameters θ, v, σ

$$E\left[e^{iuX_{NIG}(t)}\right] = E\left[\exp\left(iu\theta T_t^v - \sigma^2 \frac{u^2}{2} T_t^v\right)\right] = \exp\left[-t\left(\sqrt{v^2 - 2iu\theta + \sigma^2 u^2} - v\right)\right]$$

The NIG belongs to the family of hyperbolic distributions introduced in [6] by Barndorff-Nielsen and Halgreen who show in particular that the hyperbolic distribution can be represented as a mixture of normal densities, where the mixing distribution is a generalised inverse Gaussian density. Geman [27] emphasises that one of the merits of the expression of the stock price return as a time-changed Brownian motion

$$S(t) = W\,(T(t))$$

resides in the fact that it easily leads to the representation of the return as a *mixture of normal distributions*, where the mixing factor is conveyed by the time change, i.e. by the market activity. Loosely stated, it means that one needs to mix ‘enough’ normal distributions to account for the skewness, kurtosis and other deviations from normality exhibited by stock returns, with a mixing process which is not necessarily continuous. The mixture of normal distributions hypothesis (MDH) has often been offered in the finance literature. Richardson and Smith [47], outside any time change discussion, propose to test it by 'measuring the daily flow of information', this information that precisely drives market activity and the stochastic clock!

2.7 The CGMY Process with Stochastic Volatility

Carr, Geman, Madan and Yor [11] introduce a pure jump Lévy process to model stock prices, defined by its Lévy density

$$k_{CGMY}(x) = \begin{cases} \dfrac{Ce^{-Mx}}{x^{1+Y}} & x > 0 \\ \dfrac{Ce^{-G|x|}}{|x|^{1+Y}} & x < 0 \end{cases}$$

and show that the parameter Y characterises the activity intensity of the market to which the process is calibrated.

As any Lévy process, the CGMY process has independent increments which do not allow to capture such effects as volatility clustering which have been well documented in the finance literature. In order to better calibrate the volatility surface, Carr, Geman, Madan and Yor [12] propose to introduce in the CGMY model stochastic volatility in the form of a time change, leading to a return process

$$R(t) = X_{CGMY}(T(t)) \tag{3}$$

where the time change is meant to create autocorrelations of returns and clustering of volatility. Since the time change has to be increasing, they choose for T(t) the integral of a mean-reverting positive process, namely the square-root process:

$$T(t) = \int_0^t y(u)\,du$$

where $dy(t) = k\,(\eta - y)\,dt + \lambda\sqrt{y}\,dB(t)$.

The process described in (3) performs much better when calibrating S&P option prices through strikes and maturities (see CGMY [12]).

To conclude this section, we should observe that the representation

$$R(t) = X\,(T(t)) \tag{4}$$

where X is not necessarily Brownian motion and T, the time change, is chosen to translate the desired properties of stochastic volatility may be quite powerful if the two processes X and T are fully known, in particular in terms of trajectories. In order to price exotic options, one can build Monte–Carlo simulations of the stock process and avoid the hurdles created by the unobservable nature of volatility in stochastic volatility models.

We have shown that by changing the probability measure (and the numéraire in the economy) or changing the clock, asset price processes can be expressed as martingales or even Brownian motion. The martingale representation is immediately extended to contingent claims in the case of complete markets where there is a unique martingale measure for each chosen numéraire. In the case of incomplete markets, we are facing many martingale measures; moreover, self-financing portfolios are not in general numéraire – invariant, nor in turn the pricing and hedging of contingent claims (see Gouriéroux and Laurent [31] for the case of the minimal variance measure). These elements, among others, illustrate the numerous difficulties attached to incomplete markets. Given the importance of the numéraire-invariance property, for instance when managing a book of options involving several currencies, this feature may be a constraint one wishes to incorporate when choosing among the different answers to market incompleteness.

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Options and Partial Differential Equations

D. Lamberton

Abstract The aim of this chapter is to show how partial differential equations appear in financial models and to present succinctly numerical methods used for effective computations of prices and hedging of options. In the first part, we take up the reasoning which allowed Louis Bachelier to bring out a relationship between the heat equation and a modelling of the evolution of share prices. In the second part, the equations satisfied by options prices are introduced. The third part is dedicated to numerical methods and, in particular, to the methods based on the simulation of hazard (the so-called Monte Carlo methods). The understanding of this text does not necessitate mathematical knowledge beyond that of an undergraduate level. Thus, we hope that it may be read by non-mathematical scientists.

1 The Radiation of Probability

In his thesis (defended in 1900 in the Sorbonne; see [1,3] for more historical details), Louis Bachelier proposes a probabilistic modelling of the time evolution of the price of a share. In terms of what he calls the 'radiation of probability', he is able to relate the distribution of probability to the heat equation, which describes the evolution of temperature in a given media.

Let us denote by X_t the price of the share at time t, or rather what Bachelier calls the 'true price', which may correspond to what we would call today the *discounted price*, which allows to disregard interest rates. The starting point consists to associate to the *random variable* X_t, a *density* of probability, that is a function $x \mapsto p(t,x)$ whose integral in any space interval measures the probability that X_t belongs to that interval:

$$\mathbf{P}(a \leq X_t < b) = \int_a^b p(t,y)dy.$$

If one draws the graph of the function $x \mapsto p(t,x)$ (see Fig. 1), the probability that X_t be located between the values a and b is measured by the area of the planar region

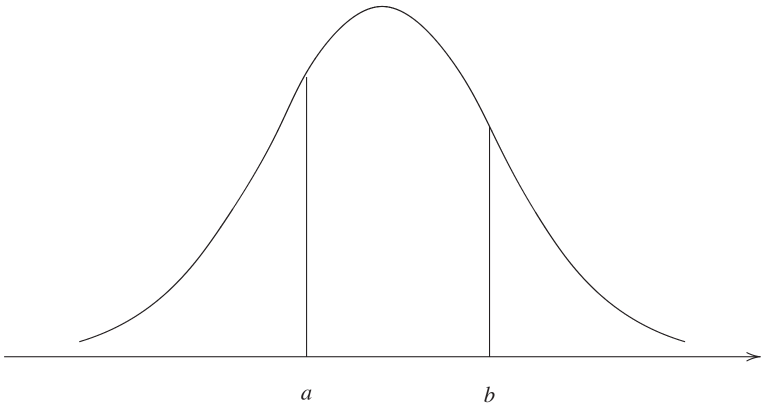

Fig. 1 The density of probability

situated under the graph between the bounds a and b. Another way to understand this notion of density consists to write (taking $a = x$ and $b = x + h$ with h small)

$$\mathbf{P}(x \leq X_t < x+h) \approx hp(t,x),$$

which makes indeed the number $p(t,x)$ appear as a *density* of probability per unit length, in the neighbourhood of the value x.

In order to reason about probabilities of events, Bachelier introduces the function F defined by

$$F(t,x) = \mathbf{P}(X_t \geq x) = \int_x^{+\infty} p(t,y)dy.$$

This function depends on two variables, the time t and the price of the share x. Bachelier's reasoning, which we detail below, leads him to a relationship between the derivative of F with respect to time and the second derivative of F with respect to the price, which is exactly the relationship between the time derivative of temperature and its second spatial derivative, that is the heat equation:

$$\frac{\partial F}{\partial t} = c^2 \frac{\partial^2 F}{\partial x^2} \ .$$

The positive constant c is a parameter which is related to the market's instability. In physics, the heat equation is deduced from a *heat balance* in the neighbourhood of a point. Bachelier's reasoning hinges on a *balance of probabilities*.

More precisely, Bachelier studies the increment $F(t+\Delta t,x) - F(t,x)$ of the function F on a small time interval $[t, t+\Delta t]$. By writing $F(t+\Delta t,x) = \mathbf{P}(X_{t+\Delta t} \geq x)$ and using the additivity property of probabilities, we obtain

$$\begin{aligned} F(t+\Delta t,x) &= \mathbf{P}(X_{t+\Delta t} \geq x \text{ and } X_t \geq x) + \mathbf{P}(X_{t+\Delta t} \geq x \text{ and } X_t < x) \\ &= \mathbf{P}(X_t \geq x) - \mathbf{P}(X_t \geq x \text{ and } X_{t+\Delta t} < x) + \mathbf{P}(X_{t+\Delta t} \geq x \text{ and } X_t < x), \end{aligned}$$

and, consequently

$$F(t+\Delta t,x)-F(t,x)=-\mathbf{P}(X_t\geq x \text{ and } X_{t+\Delta t}<x)+\mathbf{P}(X_{t+\Delta t}\geq x \text{ and } X_t<x).$$

To go further, we need to make the probabilistic model precise. Bachelier's idea is that, on a short time interval, the price may either increase, or decrease from a small quantity Δx and that, concerning the *true price*, the two events (increase or decrease) must be equally likely. Thus,

$$X_{t+\Delta t}=\begin{cases} X_t+\Delta x & \text{with probability } 1/2 \\ X_t-\Delta x & \text{with probability } 1/2 \end{cases}$$

Remarking that, under these conditions, the events $\{X_t\geq x \text{ and } X_{t+\Delta t}<x\}$ on one hand and $\{X_t<x \text{ and } X_{t+\Delta t}\geq x\}$ on the other hand reduce respectively to the events $\{x\leq X_t<x+\Delta x \text{ and } X_{t+\Delta t}=X_t-\Delta x\}$ and $\{x-\Delta x\leq X_t<x \text{ and } X_{t+\Delta t}=X_t+\Delta x\}$, we then obtain

$$F(t+\Delta t,x)-F(t,x)=-\frac{1}{2}\mathbf{P}(x\leq X_t<x+\Delta x)+\frac{1}{2}\mathbf{P}(x-\Delta x\leq X_t<x) \qquad (*)$$

Expressing the probabilities in $(*)$ with the help of the density, we obtain

$$\begin{aligned} F(t+\Delta t,x)-F(t,x) &= -\frac{1}{2}\int_x^{x+\Delta x}p(t,y)dy+\frac{1}{2}\int_{x-\Delta x}^{x}p(t,y)dy \\ &= -\frac{1}{2}\left(\int_x^{x+\Delta x}p(t,y)dy-\int_{x-\Delta x}^{x}p(t,y)dy\right). \end{aligned}$$

For Δx small, the difference between the integrals is roughly equal to the difference of the areas of the two trapezoids (see Fig. 2) and reduces to the sum of the areas of the two right triangles of width Δx and height $\frac{\partial p}{\partial x}(t,x)\Delta x$.

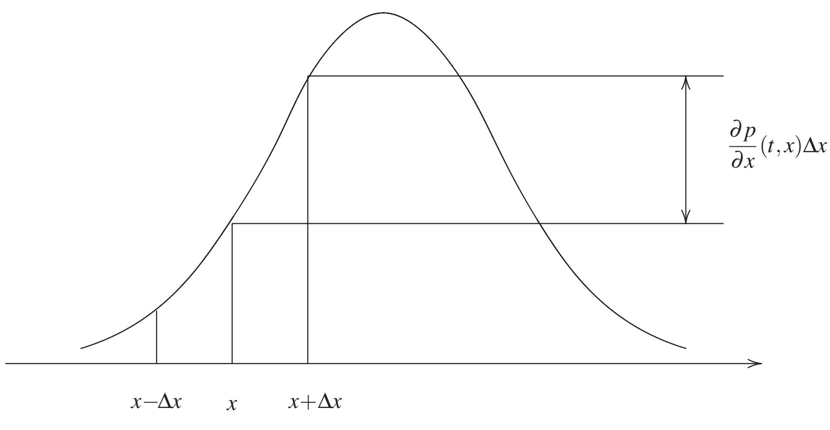

Fig. 2 The difference of the areas of the trapezoids is equal to the sum of the areas of two triangles

Hence

$$F(t+\Delta t,x)-F(t,x)\approx -\frac{1}{2}\frac{\partial p}{\partial x}(t,x)\Delta x^2,$$

and consequently

$$\begin{aligned}\frac{\partial F}{\partial t}(t,x)\Delta t &\approx -\frac{1}{2}\frac{\partial p}{\partial x}(t,x)\Delta x^2\\ &= \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,x)\Delta x^2,\end{aligned}$$

because $p=-\dfrac{\partial F}{\partial x}$. Hence, we have

$$\frac{\partial F}{\partial t}(t,x)\Delta t \approx \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,x)\Delta x^2,$$

which suggests that the function F is indeed a solution of the heat equation; this implies, by taking derivatives with respect to x, that the function p is also such a solution. The ratio $\sigma^2=\Delta x^2/\Delta t$ is related to what Bachelier calls the 'unstability coefficient', which is today called *volatility*.

The resolution of the heat equation allows to find explicitly the function p, in the form

$$p(t,x)=\frac{1}{\sqrt{2\pi t\sigma^2}}e^{-\frac{(x-m)^2}{2t\sigma^2}},$$

where m is the initial 'true price'. This density is that of the famous Gauss distribution, also called *normal law*.

Bachelier's reasoning hinges in fact on more or less implicit hypotheses found in his thesis; these hypotheses are well understood nowadays. The first hypothesis is the continuity of the trajectories of the process, which excludes any brutal variation of prices, the second one bears upon the statistical properties of the *absolute* increments of prices. The same hypothesis, formulated with respect to the *relative* increments of prices, leads to the geometric Brownian motion model which was suggested in Finance by Paul Samuelson in the 1960s, and adopted by Black–Scholes and Merton in their works on options.

2 Equations Satisfied by the Options Prices

2.1 European Options

A *European* option with maturity T is a financial asset which ensures to its owner the possibility to make a profit at date T which, in simple cases, writes $f(S_T)$, where S_T is the price, at date T, of a financial asset, called the *underlying* asset and f a positive value function, called pay-off function. Thus, in the case of a buying option on a share, at the exercise price K, which is simply the right to buy the share at the price K, one has $f(x) = (x-K)_+$ (in other words $f(x) = x-K$ if $x \geq K$ and $f(x) = 0$ if $x < K$). Indeed, if, at maturity, the share price is superior to the exercise price K, the option exercise allows, by reselling immediately the share, to make the profit $S_T - K$. In the opposite case, the option is not exercised and the pay-off is null. In a symmetric way, for a *selling* option at the exercise price K, one has $f(x) = (K-x)_+$.

The seller of the option must be able to give the owner of the option the wealth $f(S_T)$ at maturity. The problem studied by Black, Scholes and Merton at the beginning of the seventies (see [2,4]), consists to determine the value of the option across time (how much must one ask at date t for the right to obtain at a posterior date T the wealth $f(S_T)$?) to propose a hedging strategy (how must the seller of the option invest the money engaged by the buyer in order to make the expected pay-off?).

Black, Scholes, Merton's approach is based on the relationship between the price of the option and the construction of a dynamic investment strategy, allowing to reduce the risk taken by the seller of the option. In other words, one must make the buyer of the option pay the initial sum necessary to the construction of a dynamic management strategy allowing to produce the pay-off due to the owner of the option at maturity.

In order to be able to apply the rules of *stochastic calculus*, established by K. Itô in the 1950s, Black, Scholes and Merton suggest that the value of the option at date t is a function $P(t,S_t)$ depending on time t and on the price of the underlying at date t and show that the function $P : (t,x) \mapsto P(t,x)$ has to verify the partial differential equation

$$\frac{\partial P}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 P}{\partial x^2} + rx\frac{\partial P}{\partial x} - rP = 0,$$

where σ is the *volatility* of the underlying and r is the instantaneous interest rate. This equation may be seen as a variant of the heat equation. It has a unique solution, if one takes into account the terminal condition

$$P(T,x) = f(x).$$

This solution may be explicitly computed for buying and selling options: these are the famous Black–Scholes formulae.

The equation expresses in mathematical terms the fact that, up to some discounting, the variation of the value of the option on a small time interval is proportional to the variation of the price of the underlying: if $r = 0$, then:

$$dP = \frac{\partial P}{\partial x} dS.$$

Hence, the variation of the value of the option on a short period is equivalent to the pay-off made by buying at the beginning of the period and reselling at the end of the period a quantity $\partial P/\partial x$ of the underlying. Thus, Black–Scholes formulae give the price of the option (the function P) and also the hedging strategy (the function $\partial P/\partial x$).

2.2 American Options

An *American* option with maturity T may be exercised at any time before maturity, whereas the *European* options may only be exercised at maturity. If the exercise takes place at time t, the pay-off writes, in simple cases, $f(S_t)$, where S_t is the price of the underlying asset at date t and f is the pay-off function. In the case of a buying option (: call option), one has: $f(x) = (x-K)_+$, where K is the exercise price. In the case of a selling option (: put option), $f(x) = (K-x)_+$. In the Black–Scholes model, the price of the American option at date t is still a function $P_a(t,S_t)$ depending on the time and price of the underlying asset, but the function $P_a : (t,x) \mapsto P_a(t,x)$ is the solution of a more complex problem than in the European case. It satisfies two conditions which are intuitively obvious:

1. The terminal condition $P_a(T,x) = f(x)$
2. The inequality $P_a \geq f$

This inequality mirrors the fact that the option price is, at any time, at least equal to the profit allowed by the immediate exercise.

One shows, with the help of hedging arguments which are analogous to those sketched in the European case that the function P_a satisfies, moreover, the inequality

$$\frac{\partial P_a}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 P_a}{\partial x^2} + rx\frac{\partial P_a}{\partial x} - rP_a \leq 0,$$

with *equality* if $P_a > f$. It follows that the function P_a solves the same equation than the function P in the European case:

$$\frac{\partial P_a}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 P_a}{\partial x^2} + rx\frac{\partial P_a}{\partial x} - rP_a = 0,$$

but only in the domain $\{(t,x) \mid P_a(t,x) > f(x)\}$. The determination of this domain (or of its frontier) is part of the problem to be solved. One uses the term *free boundary problem*. Free boundary problems appear in other domains, in particular in physics. Among such classical problems, one may cite *Stefan's problem*, which rules the evolution of temperature in a glass of water which contains some ice cube and the *obstacle problem*, which consists in the determination of the form of a membrane covering an obstacle.

Remark. The partial differential equations appear in other problems of quantitative Finance, in particular in problems of *portfolio optimisation*. One has to determine dynamic strategies of portfolio management allowing to optimise a criteria, generally based on a *utility function* describing the *preferences* of an agent. In Mathematics, these problems result from the stochastic control theory and bring into play non-linear partial differential equations (Hamilton–Jacobi–Bellmann's equations) which have been studied extensively during the last 30 years.

3 Numerical Methods

3.1 Computational Problems in Finance

The success of the Black–Scholes' model is notably due to the fact that the price and hedging rates formulae of basic European options are explicit there and do not need much numerical computation (everything is reduced to the computation of the distribution function of the normal law, for which one has very precise approximation formulae). However, once a little complexity is added either in the option (by taking an American option for instance) or in the model (in order to remedy to the defaults of the Black–Scholes' model), then, most of the time, one has to resort to using numerical methods.

More precisely, numerical methods are needed to deal with:

- Exotic options: Besides American options, all options whose associated pay-off depends on all values of the underlying across time (and not only on the value at maturity), in particular the *Asian* options which bring into play the mean of prices during a given period.
- Complex models: The limitations of the Black–Scholes' model have led to the development of new models, in particular the models where volatility is a function of the price of the underlying (called 'local volatility models') or even a random process subjected to a noise which is partly correlated to the one directing the dynamic of the underlying (stochastic volatility models), along with the models allowing for brutal variations of prices (models with jumps); let us also cite multidimensional models allowing to deal with several underlyings at the same time, and the interest rates models.
- Calibration: One uses this term to describe the process allowing the best adjustment of the model parameters to the market data, and in particular to the prices of options quoted by the market. The model can only reasonably be used to compute option prices or hedging strategies if it gives the options quoted by the market the same price as the one given by the market.

Numerical methods used to solve partial differential equations have been developed during the second half of the twentieth century, at the same time as computing progresses. All these methods (finite differences, finite elements, wavelets, ...) make their way through the solving of quantitative Finance problems, but it is not possible to make a detailed presentation of this point throughout this article. We will rather dwell upon probabilistic methods, based on the simulation of randomness and which are most commonly called *Monte Carlo methods*.

3.2 Monte Carlo Methods

The application of these methods to compute option prices hinges on the fact that the price of a European option may be written as the mathematical expectation with respect to the so-called *risk-neutral* probability of the (discounted) pay-off allowed by the exercise of the option. The computation of the mathematical expectation of a random variable X may be done by *simulation* with the help of the **law of large numbers**, which, for a mathematician, is the theorem which asserts that the arithmetic mean of a large number of independent trials (denoted below by X_1, X_2, ..., X_N) of the random variable X converges towards the mathematical expectation of X:

$$\mathbf{E}(X) \approx \frac{X_1 + \cdots + X_N}{N} \quad (N \to \infty).$$

This method may be applied as soon as one knows how to *simulate* the random variable X, which means that one may produce, due to computer programming, the trials $X_1, \ldots, X_N$. Without getting into details of efficient simulation methods, we would like to indicate three special properties of Monte Carlo methods:

1. The law of large numbers holds under very general conditions. It suffices that the random variable be integrable, and there are no regularity conditions analogous to those necessitated by deterministic methods of numerical integration.
2. The error is of the order of $\sigma/\sqrt{N}$, where σ^2 is the variance of the random variable X. Thus, the convergence is slow: in order to divide the error by 10, one needs to multiply the number of trials by 100.
3. The method is efficient for high dimensional problems, that is when the random variable depends on a large number of independent sources of randomness, in which case the deterministic methods become unfeasible.

Let us make precise the modes of applications of the Monte Carlo methods for the computation of option prices. In simple cases, as we have already seen, the random variable X is a function of the value of the underlying at date $T : X = f(S_T)$. In the case of exotic options, X may depend on all the values taken by the underlying between the times 0 and T, so that $X = \Phi(S_t, 0 \leq t \leq T)$.

The function f (or the functional Φ) is known explicitly, but the direct simulation of S_T (or *a fortiori* of all the trajectory $(S_t, 0 \leq t \leq T)$) is not always possible. Indeed, for most models, the process S is defined as the solution of a *stochastic differential equation*

$$dS_t = b(t, S_t)dt + \sigma(t, S_t)dB_t \ .$$

Such an equation may be understood as an ordinary differential equation: the equation $dS_t = b(t, S_t)dt$ perturbated by a random noise: the term $\sigma(t, S_t)dB_t$, where B_t is a Brownian motion. For models with jumps, an additional term related to the discontinuities must be added.

In the case of the Black–Scholes model, the stochastic differential equation is solved in closed form. When this is not the case, one needs *to discretise* the equation. The discretisation of stochastic differential equations is a research topic which is very active these days, as it is stimulated by potential applications in Finance. Techniques which aim at accelerating the convergence in the Monte Carlo methods are also developed (in particular *reduction of variance* techniques).

Numerical methods based on stochastic approaches are being developed in various directions. As the mathematical expectation of a random variable depending on the whole trajectory may be seen as an integral on the (infinite dimensional) space of trajectories, techniques of numerical integration in infinite dimension begin to appear.

Recent progress often hinges upon theoretical advances of stochastic analysis. A striking example is the use of the stochastic calculus of variations due to Paul Malliavin, which may be seen as a differential calculus acting upon the set of Brownian motion trajectories. In the framework of this theory, one disposes of very useful integration by parts formulae, in particular for the computation of *sensitivities with respect to the parameters* of option prices via the Monte Carlo methods.

The sophistication of the mathematical methods employed today in Finance illustrates the progress of probability theory since the beginning of the twentieth century. At the time of Bachelier's thesis, the formal framework of probability calculus has not yet been thoroughly established. Bachelier discusses about probabilities of events which involve time dependent quantities at some fixed times. Thus, he is able to establish some link with partial differential equations which will prove to be extremely fruitful, in particular in relation with Kolmogorov's works in the thirties. The birth and development of Itô's *stochastic calculus* around the middle of the century will be a decisive step, as it will allow for a local differential time calculus along trajectories. This will be the key of the analysis of dynamical hedging strategies in the works of Black–Scholes and Merton. Today, a global differential and integral calculus on the space of trajectories begins to be usable, and already has had applications in Finance.

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Mathematics and Finance

E. Gobet, G. Pagès, and M. Yor

Abstract Since the beginning of the 1990s, mathematics, and more particularly the theory of probability, have taken an increasing role in the banking and insurance industries. This motivated the authors to present here some interactions between Mathematics and Finance and their consequences at the level of research and training in France in these domains.

1 Some History of the Subject

The origins of the mathematical foundations of modern Finance go back to the thesis of Louis Bachelier [BAC00] entitled *Théorie de la spéculation* which was defended in the Sorbonne in 1900. These works mark, in fact, the birth of stochastic processes in continuous time, and, on the other hand that of continuous time strategies for risk hedging in Finance. On the mathematical side, Bachelier's thesis greatly influenced researches of A.N. Kolmogorov about continuous time processes in the twenties as well as those of Itô – the inventor of stochastic calculus, in the 1950s. On the contrary, concerning Finance, Bachelier's approach has been forgotten for about three quarters of a century, precisely until 1973 when the works of Black, Scholes and Merton[1] ([BS73], [ME73]) appeared.

Let us consider the period of the 1970s in order to better understand the context of that era. It is at that time that the political will to deregulate financial markets emerges, which made interest rates become volatile and change rates unstable. In such a deregulated environment, the industrial and commercial companies are subjected to increased risks, which are related, for example, to the extreme variability of the change rates: this situation is uncomfortable, especially when incomes and

[1] For their works, Scholes and Merton received the Nobel prize in economics in 1997 (Black died in 1995).

expenses are labelled in different currencies (let us say dollar and euro). In order to present companies with tools adapted to these problems, and more generally to allow insurance companies and banks to cover these new kinds of risks, some newly organised markets have been created, which allowed the different parties to exchange massively insurance products. This entailed the emergence of new financial instruments, the so-called *derivative products*. The buying option (or *Call*) is the prototype of these derivatives and still remains today one of the most used instruments: in the preceding example, such an option protects the company against the rise of the exchange rate euro/dollar. Once this option is acquired today by the company, it will confer it the right, but not the obligation, to buy one dollar in exchange of K euros (the *exercise price* or *strike* K is a fixed characteristic of the contract) at the fixed future date T (called *maturity* or *expiration date*). If the exchange rate in question is S_t at date t (i.e. 1 dollar $= S_t$ euros), this insurance means for the company that it will receive an amount $\max(S_T - K, 0)$ euros at maturity T: if $S_T \le K$, the buying rate of the dollar is more advantageous than the one planned in the contract and it therefore does not perceive anything (and changes its euros into dollars on the market euro/dollar if necessary); on the other hand, if $S_T > K$, it exercises its right to acquire dollars at the most advantageous rate guaranteed by the option contract: 1 dollar $= K$ euros (how many dollars may be bought this way is also a term of the option contract).

Two questions arise then for the intervening parties on the markets: what is the price (called the *premium*) of such optional contracts, and which attitude should one adopt after selling such a product and having endorsed the risk – that of the rise of dollar against euro at maturity of the contract – in place of the buyer? Although Bachelier established, as soon as 1900 in his thesis [BAC00] the connection between the price of this type of financial instruments and some probabilistic computations relative to certain stochastic processes, the question of hedging the corresponding risk was only solved with the works of Black, Scholes and Merton in 1973. At that time, the idea of risk diversification was already fashionable, due to the pioneering works of Markowitz[2] in 1952 about portfolio optimisation: Markowitz proposes a diversification of the static risk, based upon the repartition of assets inside a portfolio. The problem is again different when it comes to damage insurance: in that case, the diversification depends on the number of insured persons. The novel approach of Black, Scholes and Merton, which still constitutes today the keystone of modern Finance consists to diversify risk *across time* (between today and maturity), by carrying out *a dynamical investment strategy*. Concerning exchange rates calls, this consists in buying or selling dollars at each instant. The *miracle* is complete when Black, Scholes and Merton show the existence of an optimal dynamical strategy, which may be computed explicitly, and which *suppresses all possible risks in all markets scenarios*.

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[2] Nobel prize in economics in 1990.

This giant's step then made it possible for these new markets to develop very quickly in an organised form. The first of them: the Chicago BOard of Trade (CBOT) opens in Chicago in 1973, quickly followed by many others, first in the United States (Philadelphia, ...), then everywhere in the world. France follows step and in 1985 creates the MATIF, meaning: Marché à Terme International de France, after several changes of signification of the acronym) then the MONEP (: Marché des Options Négociables de Paris) in 1987. Technological progress (in computers, communications, ...) as well as theoretical progress (in mathematics) also largely benefited these spectacular developments, as we shall now show.

2 The World of Black, Scholes and Merton

In order to formalise the notion of dynamical hedging, let us consider again the example of the exchange rate. At date 0, the seller receives from the buyer the premium C_0 (the price of the derivative). In time, he will invest this premium in (American) dollars. More precisely, he buys an (algebraic) quantity δ_t of them at each instant t, and the remainder is not being invested (to simplify our reasoning, we assume here that the interest rate, which remunerates the non-invested money, here euro liquidities, is zero). No import of money from outside is allowed to take place in his dynamical management: we shall say that his *portfolio is self-financed*. Denoting its value, indexed by time, as $(V_t)_{t\in[0,T]}$, then its infinitesimal variation must satisfy

$$V_{t+dt} - V_t = \delta_t \; (S_{t+dt} - S_t) \tag{1}$$

with the constraint to obtain at maturity T the amount he promised the buyer, that is

$$V_T = \max(S_T - K, 0). \tag{2}$$

At this point of the analysis, it becomes mandatory to make precise the (stochastic) model of evolution of the exchange rate $(S_t)_{t\geq 0}$. Without loss of generality, it is natural enough to decompose its instantaneous return as the superposition of a local trend and a noise. Samuelson (1960), then Black, Scholes and Merton (1973) propose to model the noise with the help of a Brownian motion $(W_t)_{t\geq 0}$, which leads to an *infinitesimal* dynamics of the type

$$\frac{S_{t+dt} - S_t}{S_t} = \mu_t dt + \sigma(W_{t+dt} - W_t). \tag{3}$$

The local amplitude of the noise is given by the parameter σ (assumed to be different from 0), called *volatility*.

In fact, the idea of introducing Brownian motion as a model for randomness in the Stock Market rates goes back to Bachelier in 1900. It is intimately linked with the very genesis of Brownian motion. This process allows to take into account in a simple manner some expected properties, such as the independence of increments, or the scaling invariance. Finally, its pathwise behaviour is quite similar to that observed in practice (see Fig. 1). However, this last point is today open to debate, and motivates investigations within larger classes of processes, such as fractional Brownian motion.

Item 1. DEFINITION OF BROWNIAN MOTION. Brownian motion is a Gaussian process, with independent and stationary increments: for $0 \leq s < t$, its increment $W_t - W_s$ follows a gaussian centred distribution, with variance $(t-s)$.
HISTORICAL SKETCH. In 1827, the botanist Robert Brown associates what shall become known as Brownian motion to the very irregular trajectories (in fact non-differentiable as functions of time) of fine particles in a fluid. In 1900, Louis Bachelier uses it as a model for the dynamics of market rates, then Einstein in 1905, to describe a diffusing particle. It is only in 1923 that Wiener formalises its mathematical construction. This date marks the beginning of intensive mathematical researches which still continue today and irrigate a large part of modern probability theory.

In fact, the rigorous justification of the infinitesimal expressions (1) and (3) is not easy, since $(W_t)_t$ has infinite total variation, but has a finite quadratic variation. Stochastic calculus, as developed by Itô in the 1950s, allows to solve these technical questions. A differential calculus may also be developed, based on Itô's formula: for every sufficiently regular function f, one has

$$f(t+dt, S_{t+dt}) - f(t, S_t) = \partial_t f(t, S_t)dt + \partial_x f(t, S_t)(S_{t+dt} - S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial^2_{x,x} f(t, S_t)dt. \qquad (4)$$

The presence of the supplementary term $\frac{1}{2}\sigma^2 S_t^2 \partial^2_{x,x} f(t, S_t)dt$, with the appearance of the second derivative of f with respect to the second variable is explained from the finite quadratic variation of Brownian motion. It is, in some sense, the trademark of stochastic differential calculus, since there is no second order term in the *ordinary* differential calculus (see Fig. 1 on next page).

With the help of these mathematical tools, Black, Scholes and Merton solve the hedging problem for Call option. Indeed, if the associated value of the portfolio writes $V_t = C(t, S_t)$, then by identifying the Eqs. (1), (2) and (4), we obtain necessarily, on one hand $\delta_t = \partial_x C(t, S_t)$ and on the other hand $\partial_t C(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{x,x} C(t,x) = 0, C(T,x) = \max(x-K, 0)$. This latter partial differential equation reduces, by change of variables, to the heat equation whose solution, which has been well-known for a long time, is explicit: thus, one obtains the famous Black and Scholes formula which is used in all market floors of the world, and gives the value $V_0 = C(0, S_0)$ of the option today. It is then remarkable that, with the above strategy,

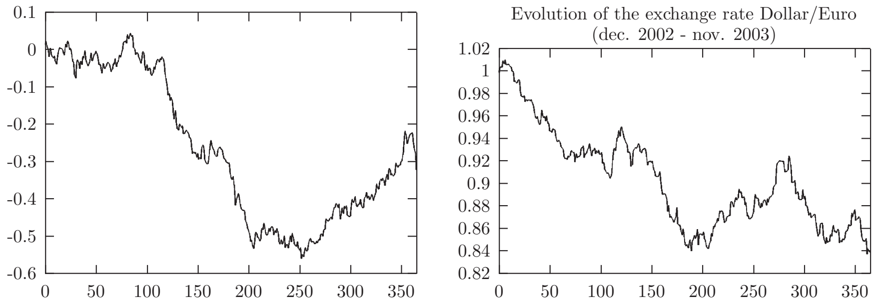

Fig. 1 Simulation of a Brownian trajectory (*on the left*) and the evolution of the exchange rate Dollar/Euro (*on the right*): there is more than a similarity, almost a family likeness...

the seller of the option is able in all markets scenarios to generate the *random target* $\max(S_T - K, 0)$. It is equally surprising to notice that the premium V_0 only depends on the model (3) via the intermediary of the volatility σ: in particular the local return $(\mu_t)_t$ does not appear in the formula.

Item 2. BLACK AND SCHOLES FORMULA. The price (or premium) (Fig. 2) of the call option of maturity T and strike K is given by the function

$$\begin{cases} C(t,x) = x\mathcal{N}[d_1(x/K,t)] - K\mathcal{N}[d_0(x/K,t)], \\ d_0(y,t) \;\; = \dfrac{1}{\sigma\sqrt{T-t}}\ln(y) - \dfrac{1}{2}\sigma\sqrt{T-t}, \\ d_1(y,t) \;\; = d_0(y,t) + \sigma\sqrt{T-t}, \end{cases}$$

where $\mathcal{N}$ denotes the cumulative function of the standard normal law.

The associated hedging strategy is given at the instant t by $\delta_t = C'_x(t,S_t) = \mathcal{N}[d_1(S_t/K,t)]$ parts of the risky asset.

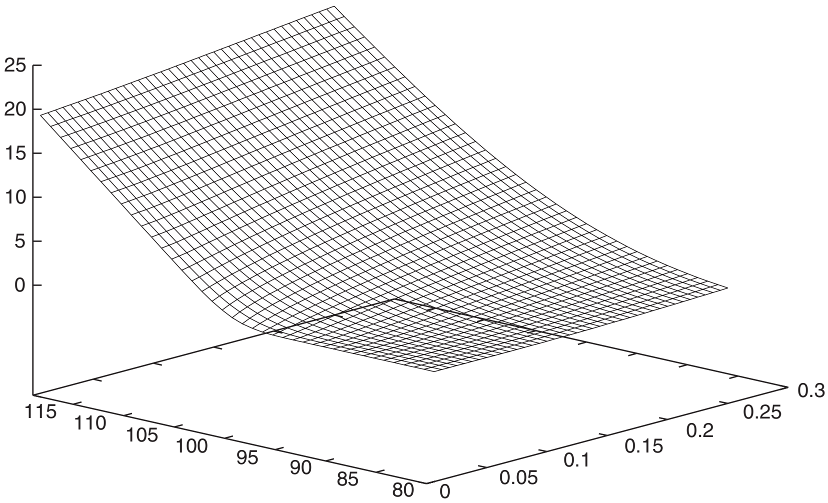

Fig. 2 The price surface as a function of x and $T-t$

It now remains to understand why the price of the option is unique. The hypothesis of *absence of arbitrage opportunity*, which is fundamental in Finance, will give the answer to this problem: this hypothesis expresses that it is not possible to win money with full certainty without any investment. Thus, concerning the call option evoked above, assuming that its price would be $V_0' > V_0$, it would suffice to sell such a contract, then, with the premium received thus, to follow the strategy $(\delta_t = \partial_x C(t, S_t))_{0 \leq t \leq T}$ in order to generate *with full certainty, and starting from nothing* a profit $V_0' - V_0 > 0$. A similar reasoning is valid for $V_0' < V_0$: it then suffices to follow the opposite strategy – in the mathematical meaning of the term – of the preceding one. This finally shows that there is only one fair price, given by the above formula. It is for this reason that the strategy δ_t is called δ*-neutral strategy*.

To end up with this subject, we indicate that the price V_0 may also be written in the form of a mathematical expectation, by using the Feynman–Kac formula which links the solution to the heat equation and Brownian motion. This writing in the form of a mathematical expectation of the value of the premium led the path to many explicit computations of option prices in the framework of the Black–Scholes model, thus bringing forth the efficiency of stochastic calculus.

3 The Practice of Markets

In the preceding section, we described the construction of a dynamical portfolio $(\delta_t)_{t\in[0,T]}$ which *simulates* or *replicates* the value of the option at its maturity, that is $\max(S_T - K, 0)$. This is at the heart of modelling in Finance. This type of situation may be found in a much more general class of models, the so-called complete markets. However, one might question in the first place the very interest of this procedure: why develop so many efforts in order to establish a formula which gives the value of an option contract at every instant of its existence, while there exists a negotiable market whose *raison d'être* is precisely to give this value through the balance of offer and demand? In fact, the answer is hidden in the approach which is adopted to establish the formula. Indeed, without entering into the working details of such a market, it is clear that, at maturity, there exist managers (or counterparts) for the option contracts which, *in fine*, will face the options holders and will need to deliver them with the famous *stochastic target* $\max(S_T - K, 0)$ euros (when $S_T > K$ obviously). Now, what will these counterparts do, between the date at which they will have received the premium (by selling an option contract) and its maturity T? Quite naturally, they will hedge, in a δ-neutral manner, as time evolves, a self-financed portfolio which consists of δ_t underlying assets S at each instant t, so that they may dispose with full certainty (therefore *without risk*) of the stochastic target $\max(S_T - K, 0)$ euros at maturity: equivalently, one also discusses about *hedging portfolios* (we do not take into account here transaction costs which constitute the payment of market actors). For this purpose, these counterparts use the explicit formula giving δ_t in the Black–Scholes model (see Item 2). The precise formula does not matter very much here; on the other hand the essential point is the presence of

σ, the volatility parameter. This parameter cannot be observed instantaneously on the market, it must then be *extracted* from it, one way or another. One such manner, which would be most natural, but is essentially ignored by the finance community, would be to estimate σ statistically from the running observation. This astatistical behaviour has undoubtedly a lot to do with the culture of the financial community. But, not only : in fact, financial analysts have much more confidence in the market than in any model. Following this belief, they *invert* the problem: the Black and Scholes formula which gives the price of the call option is a function (see Item 2) of some known parameters at time t – i.e. t, S_t, K, T – and of one unknown parameter: the volatility σ. There is no difficulty to check that the Black–Scholes formula is a function of σ, which is strictly increasing and continuous, and also one-to-one on the set of the *a priori* possible values of the option. Managers then use the market prices to extract numerically the *implied volatility*, which is the unique solution at time t of the equation

$$C(t,S_t,T,K,\sigma_{\text{impl}}) = \text{Quoted premium } (t,K,T) \quad \text{(for a quotation of the asset } S_t\text{)}.$$

Assuming that the model of the dynamics of the underlying asset were adequate, σ_{impl} would be identically equal to σ, as time t evolves, for all strikes K. In practice, things are quite different: not only does the implied volatility vary with time t, but it also varies with the strike K ! This phenomenon is known under the name of *volatility smile*, because the graph of the function

$$K \mapsto \sigma_{\text{impl}}(t,K)$$

takes approximately, for certain markets configurations, the parabolic shape of a smile (see Fig. 3).

Once the implied volatility $\sigma_{\text{impl}}(t,K)$ has been extracted from the market price thanks to a numerical inversion of the Black–Scholes formula, the manager who has sold options will adjust his hedging portfolio by acquiring for each of the options with underlying S, strike K and maturity T for which he is a counterpart, exactly $\delta_t(S_t,\sigma_{\text{impl}}(t,K))$ underlying assets on the market of S.

Seen from this angle, a market of negotiable options is thus a market of the volatility of the asset underlying the option. There, one observes the very pragmatic behaviour of agents facing the inadequacy of a model, which, on the other hand was overwhelmingly accepted from its birth due to its computational handiness. In the first place, various works have confirmed the qualitatively *robust* character of the Black–Scholes model: assuming that the volatility parameter σ is no longer constant but random, while remaining bounded below by $\sigma_{\min}$ and above by $\sigma_{\max}$, then the Black and Scholes models with parameters $\sigma_{\min}$ and $\sigma_{\max}$ bound the option prices of the general model [EKJS98]. But, such qualitative considerations could not satisfy any longer users which the 1987 krach, then the 1997 Asian crisis made each time all the more sensitive to risks which are inherent to the management of derivative products. Thus, the idea emerged to develop models with more parameters, essentially by considering the volatility σ no longer as a deterministic parameter, but as a random process, which itself is ruled by a number of parameters (see below). Then,

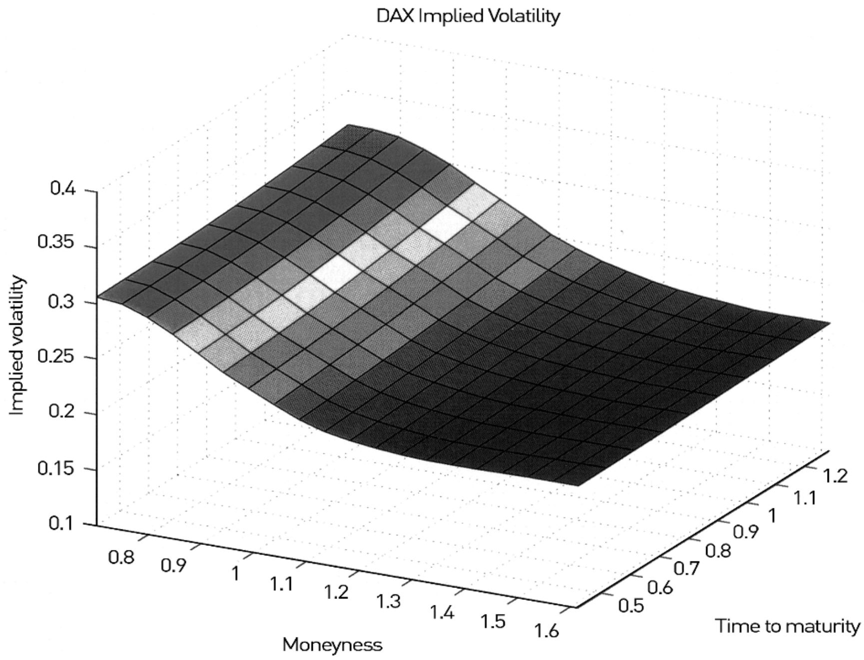

Fig. 3 Implied volatility surface as a function of K/S_0 and $T-t$ (simulation due to Cont)

they undertake to fit – financial analysts rather prefer to use the term *calibrate* – such a model with respect to the most liquid market prices by taking advantage of the greater number of degrees of freedom. This is generally done *via* the *implied volatility surface* obtained from $(K,T) \mapsto \sigma_{\text{impl}}(t,K,T)$ (which is a superposition of volatility smiles for the maturities which are present in the portfolio), this is always done by inversion of the Black–Scholes formula: one determines thus the parameters of the *enriched* model which allow to reproduce this surface as best as possible. This is a way to fit the parameters from the option prices which are quoted on the market, in a universe which is compatible with the intuition and the mode of communication of practitioners, both of them based on volatility. Once these parameters have been determined, it only remains to compute the hedges which are attached to the options contained in the portfolio and, if necessary, the non-quoted option prices, which are in general exotic (see below). It is also possible to project oneself into the future by determining, with computer simulation or with partial differential equations methods, the probabilistic structure of the portfolio, in a week or in a month, notably the probability to exit from a given frame, which was determined a priori: this is the aim of the *value at risk*, which is a financial avatar of the confidence interval [ADEH99].

In order to enrich the dynamical model of the asset, the whole probabilistic arsenal has been called for help: this has been done by adding a jump component to the dynamics of the underlying asset, by making the volatility depend upon the stock value ($\sigma = \sigma(S_t)$), by modelling the dynamics of the volatility with a diffusion process which is more or less uncorrelated from the Brownian motion W which drives the evolution of the underlying asset, by adding jumps to the volatility process, and so on. The escalation seems endless, but may be moderated due to the following: although fitting a model is made easier as the number of parameters increases, its stability is inversely proportional to this number. This is a rule which some agents learn at their expense when, from one day to the next, the fitting parameters of the volatility surface vary completely, thus destroying somewhat one's confidence in the model.

Up to now, for simplicity's sake, we only discussed the very particular case of a call option. If, from a historical view point, this framework is the one for which the theory was established (in [BS73], with as underlying asset a stock which distributes no dividend), it is only today a particularly simple example among others of a derivative product. Together with the (buying-) Call options, (selling-) Put options, followed by diverse combinations of both, the so-called: *spread*, *saddle*, *straddle*, *butterfly*, ... options, were developed, soon again to be followed by uncountably many *conditional rights* which depend not only on the stock price at maturity, but of its entire path of quotations between 0 and T : for record's sake, we cite the Asian (Call) option which is nothing else but a Call option on the mean $\frac{1}{T}\int_0^T S_t dt$ of the quotations between 0 and T, the barrier options, the options without regret, the 'cliquets', the digital options, and so on. After years of technological euphoria which made the happiness of Brownian motion virtuosos and of their students, this breaking wave of *exotic options* seems to have calmed down since the beginning of the 1990s, with the estimation and the management of risk (*value at risk*, and so on) taking the lead over the most fruitful and quite unexpected partnership between probabilists and salesmen of derivative products. Let us indicate that these exotic options do not take place in general in a negotiable market, but rather correspond to over-the-counter transactions, even for individuals through the large networks of detail banks.

If the variety of derivative products seems to be (almost) infinite, that of their supports (or underlying assets) follows suit, each of them introducing a more or less important specificity in the general approach. Besides stocks which distribute no dividend (in practice stock indexes), some new kind of options on exchange rates, on *futures* contracts,[3] on commodities, have come into play. We end up this list with a special mention of derivative products about bonds and interest rates whose common underlying is in some sense the interest rates curve across maturities. This is a domain which has an essential economical importance because of the large volumes which are exchanged there; as well, it is a domain of great complexity in terms of modelling, since one has to take into account variations not only for one stock, or for a basket of them, but for a (quasi-)continuum of rates (up to 1 day, 1 month, 3 months, 1 year, 3 years, ..., 30 years) which vary randomly between them and across time, in a more or less correlated manner, thus describing a kind of random process taking values in a space of functions. Among the domains which emerged

[3] Negotiable futures contracts which made the success of the Parisian MATIF until the middle of the 1990s. Today, they are also traded in Frankfurt (BUND) and in London (LIFFE).

or are still developing, let us mention questions related to portfolio optimisation, transaction costs, *hedge funds*, ... Recently, *credit risk* or *default risk* and different related products have taken a fundamental importance: the objective is to protect oneself against the risk associated with bonds – unpaid coupons, partial or total loss of capital – issued by a company which may go bankrupt.

We now discuss another *qualitative variable* which is specific to the world of options: the range of exercise rights; so far, we discussed implicitly the so-called *European* contracts which give to their holder the right to receive *at date* T a monetary flow equal to $\max(S_T - K, 0)$. If this right is extended to the whole time interval $[0,T]$, that is one may exercise *once, and only once* at a date t of one's own choice the right to receive $\max(S_t - K, 0)$, one then speaks of an *American* option. There, one faces an optimal stopping problem which implies, for the contract holder to take a decision in a random environment. Most of the organised option markets on stocks or indexes deal with American options.

4 Mathematical Developments

The development of the markets dedicated to derivative products in the 1970s and 1980s, as well as the crises which shook them in several instances, have greatly contributed to the blossoming and development of several branches of applied mathematics, and in the first place, of probability theory and stochastic calculus; in this respect, the most striking fact appears to be the sudden shift of Brownian motion, Itô's formula and stochastic differential equations from the closed circle of the purest of probabilists...to business schools! As far as mathematics is concerned, such an example is not unique but, in the recent era, it is particularly spectacular. Another notable specificity is related to the very nature of financial markets: it is a human activity which evolves in constant urgency and is in permanent mutation; there, modelling has, at the same time, a central and volatile position: what is true today may not be so tomorrow. The mathematician, who, by nature, is eager to solve problems, may find interesting questions there whereas, on the other side, every financial analyst is eager to obtain solutions! However, both groups may face some disillusions for, when the mathematician shall be attached, above all, to finding a rigorous and exhaustive resolution of the problem he was asked for, the financial analyst shall prefer the *interpretability* of models and of their parameters (in order to get a mental representation of the universe of possible decisions) and above all the ease of implementation (explicit formulae, numerical performances, ...) which is the only way to preserve his reactivity in the midst of delicate transactions (where the unity of time is the second).

The domain where the interaction with Finance has been the strongest is clearly that of probability theory: stochastic calculus and Brownian motion in a first period, notably with the emergence of exotic options. Then, little by little, the increasing complexity of products, the escalation of models, the multiplication of indicators necessary to delimitate the sources of risks, have led to situations where explicit computations need, at least partially, to give way to numerical methods. Two large families of methods are available, those originating from numerical analysis and those from numerical probability. Each of these two disciplines may be summarised in almost one word: partial differential equations for one, Monte Carlo method for the other (that is, the computation of a mean by massive computer simulation of random scenarios). Numerical analysis, the historical pillar of applied mathematics in France, found a new source of problems where its methods, with well-tested efficiency, might be implemented. On the other hand, numerical probability, under the impulse of quantitative Finance, has undergone an unprecedented development notably with the methods of processes discretisation (in particular with the role of Denis Talay in Inria). Most of the essential domains of probability theory are put to contribution, including the Malliavin calculus (: the calculus of stochastic variations), which came recently to play an important, although in some respect, unexpected, role. Other domains of probability theory have known a truly rejuvenating period, in particular the optimal stopping theory via American options, or optimisation theory which plays an overwhelming role, from the Föllmer–Sondermann mean–variance hedging theory to the many calibration algorithms. However, the development of probability and of simulation was not detrimental to other more theoretical aspects since, during the last several years, jump processes, which have been more usually associated with queuing and networks problems, have also been used today massively in financial modelling, generally in their most sophisticated aspects (*Lévy processes*, see for example [CT04]).

Finally, as is often the case in this type of interaction, financial modelling led to the emergence of new problems which developed essentially in an autonomous manner within probability theory: this is notably the case for questions arisen from the generalisation of the notion of arbitrage, either to spaces of more and more general processes, or to more realistic modelling of markets activities (taking into account the bid-ask spread on quotations, and discussing various bounds about managers' degrees of freedom, and so on).

5 Training

The development of Mathematical Finance in the 1980s has had a strong impact in terms of training in applied mathematics, mainly in probability theory, under the initial impulse of Nicolas Bouleau, Nicole El Karoui, Laure Élie, Hélyette Geman, Jean Jacod, Monique Jeanblanc, Damien Lamberton, Bernard Lapeyre. Since the end of the 1980s, the first courses of stochastic calculus oriented towards Finance appear, notably in the École nationale des Ponts et Chaussées, then quickly in École Polytechnique. It is remarkable that universities also took an important role in these developments, in particular on the campus of Jussieu, as at the same time, courses specialised in Finance were taught within the probabilistic DEA's (= graduate studies courses) of the Paris VI and Paris VII universities. Success has been immediate, and so remained throughout the years: while the first promotion of the specialty

Probabilités & Finance within the DEA of Probabilités et Applications of the University Paris VI (in collaboration with Ecole polytechnique concerning the Finance theme) numbered only five graduates in 1991, after 2003 each promotion numbers in general more than 80 graduates. A similar dynamic is observed within University Paris VII. Meanwhile, the 'old' terms of DEA or DESS have been changed into M2 (= second year of Master studies), with respective qualifications of *Research* and *Professional*. Today, if we consider only the extended Paris region (=Ile-de-France), and only these trainings, three other Masters oriented towards Mathematical Finance have developed successfully: the DEA *Mathématiques Appliquées aux Sciences Économiques* of Paris IX (which has become Master 2 MASEF) and the Master 2 *Analyse & Systèmes aléatoires* (major in Finance) at the University of Marne-la-Vallée, the DESS *Ingénierie financière* at Evry-Val-d'Essonne. The students engaged in these different trainings benefit from the strong points of the different local research teams (modelling, stochastic calculus, numerical probability, econometry, statistics ...). One may estimate roughly that between 150 and 200 students graduate each year via these 'Paris region' trainings (which often work in partnership with Engineering and/or Business Schools and welcome many students from these Institutions, who are looking for a top level training in Mathematical Finance). One may thus evaluate that about 15% of the students from Ecole polytechnique access the different professions of quantitative Finance via their mathematical knowledge and ability. Besides Ecole polytechnique, many specialised trainings in Mathematical Finance are blossoming within Engineering Schools, such as ENPC, ENSAE, Sup'Aéro in Toulouse or Ensimag in Grenoble, which is very much ahead in this domain.

Further than their specificities and their professional and/or academic orientations, these trainings are a must for the future quant (*analystes quantitatifs* in French). They are organised around three main directions: modelling (based essentially on stochastic calculus), probability and numerical analysis, optimisation, algorithmic and computer programming (see [EKP04] for further details). One must take into account that the research cells of banks in which many *quants* begin their carriers, function generally as providers for other services of their institution (trading room, manager, ...). Thus, they function like small structures of the PME (acronym for French: petites et moyennes entreprises – small businesses) type. This is all the more true in institutions with relatively small sizes (managing societies, funds, etc). Thus, there is a real necessity to be versatile.

The impact of Mathematical Finance may also be observed within fields which are not primarily mathematically oriented. This is notably true in older training programs which generally end up studies in economics or management (DEA *Banques & Finance* in Paris 1, *203* in Paris 9, actuaries trainings as that of Lyon II or ENSAE, ...); likewise, in Business Schools such as HEC or ESSEC, mathematics for Finance often takes a significant place. This illustrates the position taken by the applied mathematics culture in *a priori* less *quantitative* domains, such as management or sales, trading of assets or financial products.

If France takes an important position in the training of *quants*, which is due in a large part to the importance traditionally given to mathematics in the training of young French people, the employment possibilities in Market Finance are obviously directly related with the importance of financial places. Today, the Europe of Finance and the employment which goes with it develop essentially in London where, each year, an increasing part of young graduates will put in practice there the theoretical know-how acquired in the Hexagone. London is not, far from that, their only destination: many of them do not hesitate to answer the call of the wide world, and go earn their spurs in New York or Tokyo, ...

The attraction capacity of these trainings for talented foreign students is quite obvious, given this most favourable context. However, it may be slowed down due to the language barrier and some adaptability to the French system, notably concerning the mode of evaluation, which, quasi exclusively, remains the resolution of problems in limited time, a criterion which is far from being universally adopted throughout the rest of the world, and for which foreign students are often ill-prepared. Conversely, the almost cost free French university system should, if it lasts, constitute a major advantage. This is a French specificity, which may seem quite astonishing in comparison with both Great Britain and United States similar trainings, for which the cost often reaches the equivalent of several tens of thousands of euros.

As a conclusion, let us note that the message has diffused among the upcoming generations and that one observes more and more undergraduate students, and students in Engineering Schools which take up – sometimes with difficulty – advanced mathematical studies with the unique goal to access the jobs of Market Finance. Whether one rejoices or deplores this fact, stochastic calculus, and by extension, probability and applied mathematics have become, during the last 15 years *the* access road in the scientific field to the jobs of Market Finance. At the moment, it is a 'highway', future shall tell ...

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Author Index

Subject Index